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Technical Report

An Equivalent-Circuit Model for Flexural-Disk Transducers

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An Equivalent-Circuit Model for Flexural-Disk Transducers

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The goal of this project was to develop a set of analysis tools for performance prediction of flexural-disk acoustic projectors. The emphasis is on analytically based formulations designed to merge the physics of the flexural-disk structure with typical operational requirements of frequency, source level, bandwidth, and operating depth. While finite element analysis coupled with acoustic radiation models would be used to refine designs, analytical models permit isolation of the major effects of essential design variables. By developing an understanding of the transducer performance, the effectiveness of various strategies for flexural-disk designs can be evaluated. The model described in this report covers both center-supported and edge-supported structures and two- and three-layer configurations.

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Abstract

The goal of this project was to develop a set of analysis tools for performance prediction of flexural-disk acoustic projectors. The emphasis is on analytically based formulations designed to merge the physics of the flexural-disk structure with typical operational requirements of frequency, source level, bandwidth, and operating depth. While finite-element analysis coupled with acoustic radiation models would be used to refine designs, analytical models permit isolation of the major effects of essential design variables. By developing an understanding of the transducer performance, the effectiveness of various strategies for flexural-disk designs can be evaluated.

The model described here covers four configurations of flexural-disk transducer element. These are two- and three-layer disks clamped at the center and two- and three-layer disks simply supported at the edge. One of the layers in each case is a passive substrate, the other layer(s) is (are) piezoelectric ceramic. The only constraints on the layer thickness are that the three-layer configurations must have the same thickness of ceramic on both sides of the substrate and that in all configurations the overall thickness be small with respect to the radius. For the center-clamped case, a non-zero radius for the clamping post is permitted (required, in fact, since the zero-radius center clamp is unrealistic). An option is included to permit treatment of a fluid-filled volume behind the disk.

The equivalent-circuit model is constructed including dielectric and mechanical losses and it also includes a simple model for the radiation load. Using this equivalent-circuit model, the driving-point admittance (with and without water load), the transmitting voltage and current responses, and the free-field voltage receiving response are calculated. With the file structure employed in the MathCad program, the user can export the equivalent-circuit parameters to other applications or the user can import equivalent-circuit parameters from other sources to run the internal performance calculations.

Acknowledgments

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I. Introduction

This report describes the development of an equivalent circuit model for flexural disk transducers[1]. Four configurations are considered – two center-supported structures (Figs. 1 and 2) and two edge-supported structures (Figs. 3 and 4).

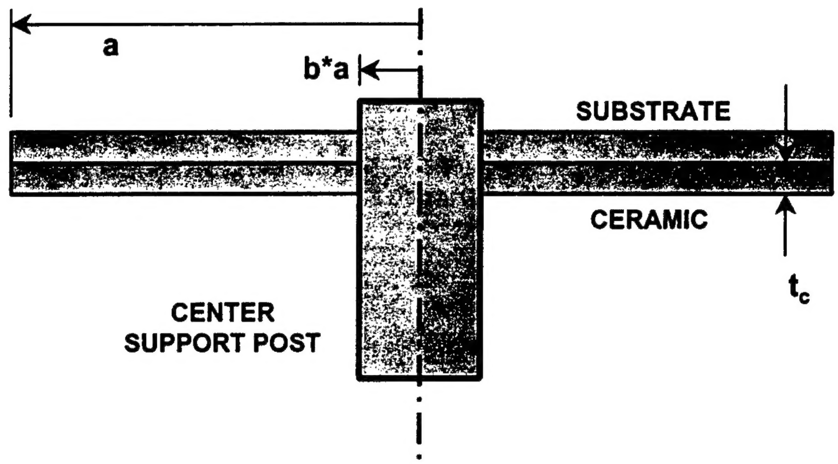

Fig. 1. Two-layer, center-supported flexural disk structure. In this view, the water would be on the upper face and the lower face would contact a gas back-volume. For many practical transducers, the structure would be doubled so as to have two structures back-to-back. The edges are nominally free but would, in practice, be sealed with boots or bellows.

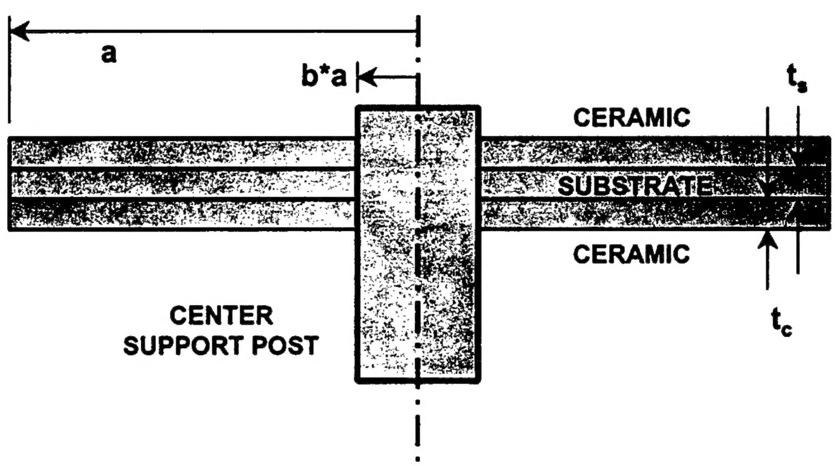

Fig. 2. Three-layer, center-supported flexural disk structure. This structure is assumed to be symmetric with equal thickness of ceramic on either side of the passive substrate.

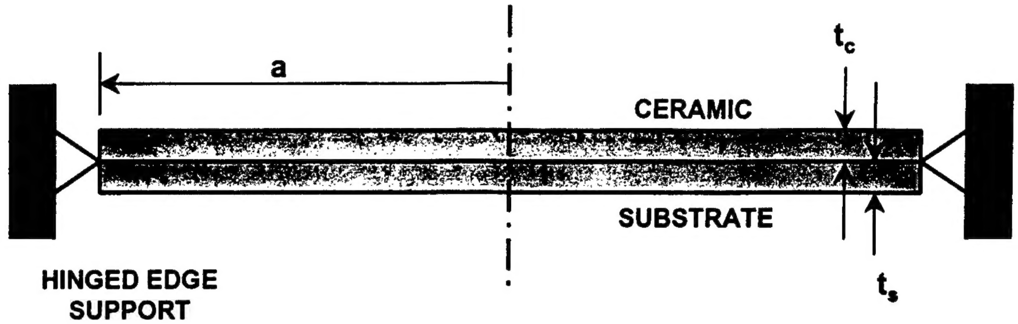

Fig. 3. Two-layer, edge-supported flexural disk structure. As in Fig. 1, water would contact the upper surface. For designs not compensated for hydrostatic pressure, this would ensure that the ceramic is in compression. For the edge-supported transducers, the housing contacts the active element at its circumference. The edge is simply supported.

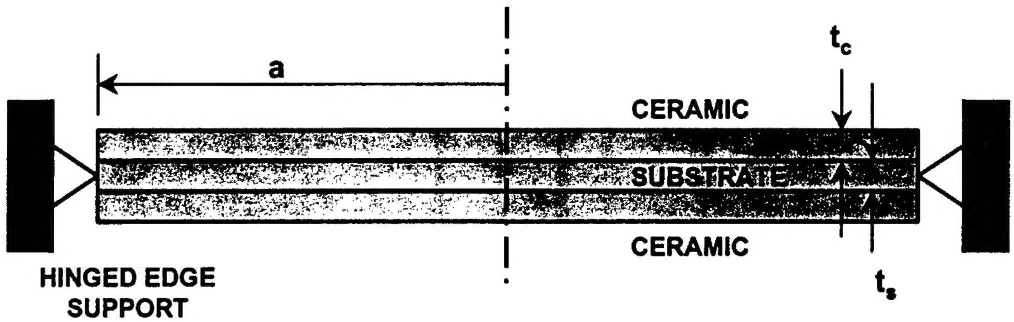

Fig. 4. Three-layer, edge-supported flexural disk structure. As in Fig. 2, only the symmetric case with both ceramic disks equal in thickness will be considered.

Matrix notation is used throughout. In order to provide continuity with the bulk of the transducers literature, electric flux density, D, is used to combine the effects of free space and materials properties (through the permittivity, ε). A better approach is to treat all dielectric properties through the polarization, P, and susceptibility, χ. As long as the material is strictly linear, use of the older convention will not cause problems but, for high drive levels, a formulation in terms of polarization may be of more value[2,3]. Also, thin-plate theory[4] will be used; however, the approach to including shear will be described briefly.

II. Basic Piezoelectric Equations

To solve the flexural-disk problem, the plane-stress assumptions are invoked and the matrix equations can be reduced significantly. The basic assumptions of plane-stress are:

- Normal stress in poling direction (T_z or T_3) is negligible.
- "Vertical" shear stresses (T_{xz}, T_{yz} or T_4, T_5) are negligible.

Furthermore, from the electrode configuration (electrodes only on disk surfaces, not edges), the electric field is assumed to be significant only in poling direction (E_z or E_3).

The basic equations that couple the mechanical and piezoelectric behavior relate four variables: the strain, S, the stress, T, the electric field, E, and the electric flux density, D. In three dimensions, each of these quantities is a matrix. Since the plane-stress assumptions constrain stress not strain, the following general form[5] is the appropriate starting point:

$$S = s^E T + d^{tr} E \tag{1}$$

$$D = d\,T + \varepsilon^T E \tag{2}$$

which can be expanded without further approximation as

$$S_1 = s_{11}^E T_1 + s_{12}^E T_2 + d_{31} E_3 \tag{3}$$
$$S_2 = s_{12}^E T_1 + s_{11}^E T_2 + d_{31} E_3 \tag{4}$$
$$D_3 = d_{31} T_1 + d_{31} T_2 + \varepsilon_{33}^T E_3 \tag{5}$$

This is a complete description of the in-plane mechanical behavior and the electrical behavior. The out-of-plane strain, S_3, is not zero but that strain does not enter our analysis. If we had chosen an equation set with S as an independent variable, then S_3 could be eliminated using the $T_3 = 0$ equation.

In some circumstances, other forms of these equations are more convenient. If we treat Eqs. 3-5 as a single matrix equation[6],

$$\begin{bmatrix} S_1 \\ S_2 \\ D_3 \end{bmatrix} = \left[\begin{array}{cc|c} s_{11}^E & s_{12}^E & d_{31} \\ s_{12}^E & s_{11}^E & d_{31} \\ \hline d_{31} & d_{31} & \varepsilon_{33}^T \end{array}\right] \begin{bmatrix} T_1 \\ T_2 \\ E_3 \end{bmatrix} \tag{6}$$

where the 3x3 matrix on the right-hand side has been partitioned to show its composition. Inverting this matrix equation leads directly to the equivalent coefficients for the plane-stress approximation:

$$\begin{bmatrix} T_1 \\ T_2 \\ E_3 \end{bmatrix} = \left[\begin{array}{cc|c} cc_{11}^D & cc_{12}^D & -hh_{31} \\ cc_{12}^D & cc_{11}^D & -hh_{31} \\ \hline -hh_{31} & -hh_{31} & \beta\beta_{33}^S \end{array}\right] \begin{bmatrix} S_1 \\ S_2 \\ D_3 \end{bmatrix} \tag{7}$$

The effective stiffness coefficients (cc), piezoelectric coefficient (hh), and inverse permittivity ($\beta\beta$) can be extracted directly from the inverse of the original 3x3 matrix. In order to emphasize the difference between the two- and three-dimensional coefficients, those coefficients having different values in two dimensions will be denoted by doubling the letter. For example, the

stiffness coefficients for the reduced two-dimensional problem will be denoted cc_{11}, and cc_{12}. In setting up these disk problems, manufacturers' values for the coefficients can only be used in Eq. 3. The other coefficients must be derived through the matrix inverse. The matrix equation, Eq. 7, contains the three equations,

$$T_1 = cc_{11}^D S_1 + cc_{12}^D S_2 - hh_{31} D_3 \tag{8}$$

$$T_2 = cc_{12}^D S_1 + cc_{11}^D S_2 - hh_{31} D_3 \tag{9}$$

$$E_3 = -hh_{31} S_1 - hh_{31} S_2 + \beta\beta_{33}^S D_3 \tag{10}$$

Also, in polar coordinates, the strain matrix is

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} S_{rr} \\ S_{\theta\theta} \end{bmatrix} \tag{11}$$

with an equivalent form for the stress matrix.

Summary of Conversion to Two-Dimensions

In order to construct the plane-stress equations, the following subset of the three-dimensional quantities are required:

$s^E{}_{11}$, $s^E{}_{12}$, d_{31}, and $\varepsilon^T{}_{33}$.

These permit construction of the matrix equation, Eq. 6. Inverting the 3x3 matrix in Eq. 6 produces the values for $cc^D{}_{11}$, $cc^D{}_{12}$, hh_{31}, and $\beta\beta^S{}_{33}$.

To illustrate the difference between the effective plane-stress quantities and the three-dimensional quantities, consider PZT-8. Table I compares the 2-D and 3-D values for the stiffnesses, the h_{31} coefficient, and the permittivity at constant strain.

Table I. Comparison of properties[7] for three-dimensional analysis and two-dimensional (plane-stress) analysis for PZT8.

Property	3-D	2-D	[units]
$s^E{}_{11}$	11.5	same	x 10^{-12} m^2/N
$s^E{}_{12}$	-3.7	same	x 10^{-12} m^2/N
d_{31}	-97	same	x 10^{-12} C/N
$\varepsilon^T{}_{33}/\varepsilon_0$	1000	same	-
$c^D{}_{11}$	147	97	x 10^9 N/m^2
$c^D{}_{12}$	86.8	55.2	x 10^9 N/m^2
h_{31}	-0.78	-1.93	x 10^9 V/m
$\varepsilon^S{}_{33}/\varepsilon_0$	561	727	-

A point to note regarding two-dimensional analysis of piezoelectric materials concerns the Poisson's ratio, σ. There are two values for σ – the value for constant electric field, σ^E, and the value for constant electric flux density (or constant polarization), σ^D. For either value, the Poisson's ratio in the 12-plane is

$$\sigma = \frac{-s_{12}}{s_{11}} = \frac{c_{12}}{c_{11}} \tag{12}$$

Because the piezoelectric material is anisotropic, the Poisson's ratio in the 12-plane can be larger than 0.5. The Poisson's ratio in either the 13- or 23-plane is considerably smaller so the material is still volumetrically stable. For PZT-8 (see Table I), the value of σ^D derived from the 3-D properties is about 0.55. The corresponding value from the 2-D properties is 0.46 but, for some materials, the σ^D from the 2-D properties is also greater than 0.5.

Table II presents a summary of the appropriate two-dimensional properties for several generic piezoceramic types[7].

Table II. Properties for typical piezoceramic materials. Properties with doubled letters are those values corresponding to plane-stress conditions.

Property	PZT4	PZT5H	PZT8	[units]
s^E_{11}	12.3	16.5	11.5	x 10^{-12} m^2/N
s^E_{12}	-4.05	-4.78	-3.7	x 10^{-12} m^2/N
d_{31}	-122	-274	-97	x 10^{-12} m^2/N
$\varepsilon^T_{33}/\varepsilon_0$	1300	3400	1000	-
ρ	7600	7500	7500	kg/m^3
s^D_{11}	11.0	14.0	10.4	x 10^{-12} m^2/N
s^D_{12}	-5.34	-7.27	-4.76	x 10^{-12} m^2/N
cc^D_{11}	119	97.8	121	x 10^9 N/m^2
cc^D_{12}	57.7	50.8	55.2	x 10^9 N/m^2
hh_{31}	-1.87	-1.35	-1.93	x 10^9 V/m
$\varepsilon\varepsilon^S_{33}/\varepsilon_0$	892	1950	728	-

III. Relationship of Strain to Deflection

In order to analyze the behavior of flexural-disk structures, a deflection function is determined. For thin-plate problems, the exact solution can be found for both static deflection and for dynamic deflection at resonance. Alternatively, approximate forms can be developed for the

deflection function and optimized by minimizing the strain energy (for static deflection) or by minimizing the resonance frequency (for resonance deflection). Once the deflection function is determined, the strains can be evaluated.

In polar coordinates, if the deflection function is $w(r, \theta)$, then

$$S_1 = S_{rr} = -z\left(\frac{\partial^2 w}{\partial r^2}\right) \tag{13}$$

$$S_2 = S_{\theta\theta} = -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \tag{14}$$

$$S_6 = S_{r\theta} = -2z\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \tag{15}$$

where z is the displacement (in the 3-direction) away from the neutral plane.

In general, flexural-disk transducers will exploit only axially symmetric modes. In those cases, the strains are:

$$S_1 = S_{rr} = -z\left(\frac{\partial^2 w}{\partial r^2}\right) \tag{16}$$

$$S_2 = S_{\theta\theta} = -z\left(\frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{17}$$

and S_6 is zero.

IV. Basic Solution Process

The analysis starts with determination of the deflection function. For simple, homogeneous disk problems an exact solution[8] for deflection can be done (see Appendix A). The Version 3 model uses an approximate technique (see Appendix B) to permit flexibility in future revisions. Once the deflection function has been determined, the strain is derived from Eqs. 16 and 17. Then the basic piezoelectric equation set with strain as an independent variable (Eqs. 8-10) can be solved.

Solution of the basic piezoelectric equation set proceeds as follows:

A. Integrate (10) over z (thickness) and solve for D_3. Because there are no electrodes in the z-direction (i.e., on the edges of the plate), D_3 is not a function of z. The electric field, E, is a function of z; however, it is not necessary to determine that dependence. The integral of E over thickness is identical to the electrode voltage, which is the quantity of importance.

B. Integrate this expression for D_3 over the electrode area. The electrode voltage is constant over the electrode. The integral of D over the electrode area is the total charge. Set the charge equal to zero and solve for the open-circuit voltage, V^{OC}.

C. The result produced in A gives D_3 as a function of electrode voltage and the strains. D_3^{OC} results from setting $V = V^{OC}$; D_3^{SC} results from setting $V = 0$.

D. The potential energy density for strain energy is $\frac{1}{2} S_1 T_1 + \frac{1}{2} S_2 T_2$. The open-circuit potential energy density is written using (8) and (9) with D_3^{OC}; the short-circuit potential energy density is written with D_3^{SC}.

E. The strains, S_1 and S_2, can be written in terms of the deflection function according to Eqs. 16 and 17.

F. Integrate the potential energy densities over the ceramic volume to obtain U^{OC} and U^{SC}.

G. Integrate D_3^{SC} over the electrode area to obtain the short-circuit charge, q^{SC}.

H. Integrate the deflection, w, over the face area; multiply by ω, and divide by the face area to obtain the average face velocity, v_{avg}.

I. Integrate the kinetic energy density, $\frac{1}{2} \rho v^2 = \frac{1}{2} \rho \omega^2 w^2$, over the ceramic volume to obtain the kinetic energy, KE.

J. The blocked dielectric energy density is $\frac{1}{2} (D_3 E_3)^{S=0}$. Use Eq. 10 to write this energy density and integrate over the volume of the ceramic to obtain the blocked dielectric energy, U^{BD}.

K. Determine the basic equivalent-circuit parameters –

$$m = \frac{\partial^2 KE}{\partial v_{avg}^2} \tag{18}$$

$$k = \frac{\partial^2 U^{SC}}{\partial w_{avg}^2} \tag{19}$$

$$C_B = \frac{\partial^2 U^{BD}}{\partial V^2} \tag{20}$$

$$\phi = -\frac{\partial q^{SC}}{\partial w_{avg}} \tag{21}$$

where m and k are the lumped mechanical mass and stiffness, respectively; C_B is the blocked capacitance; and ϕ is the electromechanical conversion ratio (current per velocity or charge per displacement).

L. Calculate the electromechanical coupling factor:

$$\kappa^2 = \frac{U^{OC} - U^{SC}}{U^{OC}} \tag{22}$$

M. Calculate the resonance frequency:

$$\omega^2 = \frac{k}{m} \tag{23}$$

V. Solution Details

This section will treat only the ceramic layer. The equations are complicated enough that including the arbitrary neutral plane location and the substrate dynamics will be deferred to Appendix C and Appendix D as referenced at the end of this section.

A. Find expression for electric flux density, D_3, in terms of the electrode voltage by integrating Eq. (10) over z. For now, assume that the neutral plane is on the inner surface of the ceramic.

$$V = \int_0^t E_3\,dz = -hh_{31}\int_0^t (S_1 + S_2)\,dz + \beta\beta_{33}^S\,t\,D_3 \tag{24}$$

Solving for D_3:

$$D_3 = \frac{\varepsilon\varepsilon_{33}^S}{t}\left[V + hh_{31}\int_0^t (S_1 + S_2)\,dz\right] \tag{25}$$

B. Find the open-circuit voltage by integrating Eq. (24) over the electrode area and setting the surface charge equal to zero:

$$V \cdot A = -hh_{31} \int_A \int_0^t (S_1 + S_2) dz\, dA + \beta\beta_{33}^S t \int_A D_3\, dA \tag{26}$$

The integral of D_3 over area (last term on right side) is the surface charge. Set this equal to zero to obtain the open-circuit voltage:

$$V^{OC} = -\frac{hh_{31}}{A} \int_A \int_0^t (S_1 + S_2) dz\, dA \tag{27}$$

C. Find D_3^{OC} and D_3^{SC} by setting $V = V^{OC}$ and $V = 0$ respectively in Eq. (25).

$$D_3^{OC} = \frac{\varepsilon\varepsilon_{33}^S}{t} \left[V^{OC} + hh_{31} \int_0^t (S_1 + S_2) dz \right] \tag{28}$$

$$D_3^{SC} = \frac{\varepsilon\varepsilon_{33}^S hh_{31}}{t} \int_0^t (S_1 + S_2) dz \tag{29}$$

D. Write expressions for the strain energy density, $\widetilde{U}$, for both the open- and short-circuit electrical conditions.

$$\widetilde{U} = \frac{1}{2} S_1 T_1 + \frac{1}{2} S_2 T_2 \tag{30}$$

Use Eqs. 8 and 9 to replace T_1 and T_2:

$$\widetilde{U}^{OC} = \frac{1}{2} \left\{ cc_{11}^D S_1^2 + 2 cc_{12}^D S_1 S_2 + cc_{11}^D S_2^2 - hh_{31} D_3^{OC} (S_1 + S_2) \right\} \tag{31}$$

$$\widetilde{U}^{SC} = \frac{1}{2} \left\{ cc_{11}^D S_1^2 + 2 cc_{12}^D S_1 S_2 + cc_{11}^D S_2^2 - hh_{31} D_3^{SC} (S_1 + S_2) \right\} \tag{32}$$

E. Replace S_1 and S_2 in the above equations by their equivalents in terms of the deflection function (Eqs. 16 and 17):

$$S_1 \quad = \quad -z\left(\frac{\partial^2 w}{\partial r^2}\right) \quad = \quad -z\,w'' \tag{33}$$

$$S_2 \quad = \quad -z\left(\frac{1}{r}\frac{\partial w}{\partial r}\right) \quad = \quad -\frac{z}{r}\,w' \tag{34}$$

where the primes indicate differentiation with respect to r. Integration of S_1 or S_2 with respect to z from 0 to t is equivalent to replacement of z by $t^2/2$. Similar integration of either quantity squared (or S_1S_2) is equivalent to squaring (or performing the cross product), then replacing z^2 by $t^3/3$. (Note: this is where the assumption that the neutral plane is at the ceramic-substrate interface is made explicit; $z = 0$ is the location of the neutral plane (one side of the ceramic) and $z = t$ is the other side of the ceramic.) For example,

$$V^{OC} \quad = \quad \frac{hh_{31}\,t^2}{2A}\int_A \left(w'' + w'/r\right) dA \tag{35}$$

$$D_3^{SC} \quad = \quad -\frac{\varepsilon\varepsilon_{33}^S\,hh_{31}\,t}{2}\left(w'' + w'/r\right) \tag{36}$$

$$D_3^{OC} \quad = \quad \frac{\varepsilon\varepsilon_{33}^S\,hh_{31}\,t}{2}\left[\frac{1}{A}\int_A \left(w'' + w'/r\right) dA \;-\; \left(w'' + w'/r\right)\right] \tag{37}$$

F. Integrate the strain energy densities over the ceramic volume to obtain the total strain energies. First, integrate over z from 0 to t:

$$\int_0^t \tilde{U}^{SC}\,dz \quad =$$

$$\frac{1}{2}\left\{\frac{t^3}{3}cc_{11}^D\left[\left(w''\right)^2 + \left(w'/r\right)^2\right] + \frac{t^3}{3}cc_{12}^D\,2\,w''\,w'/r \;+\; hh_{31}\,D_3^{SC}\,\frac{t^2}{2}\left(w'' + w'/r\right)\right\} \tag{38}$$

or,

$$\int_0^t \tilde{U}^{SC}\,dz \quad = \quad \frac{t^3}{6}\left\{f_{11}\left[\left(w''\right)^2 + \left(w'/r\right)^2\right] + f_{12}\left[2\,w''\,w'/r\right]\right\} \tag{39}$$

where the f parameters have been introduced to simplify the notation:

$$f_{11} \;\equiv\; cc_{11}^{D} \;-\; \frac{3}{4}\varepsilon\varepsilon_{33}^{S}\,hh_{31} \qquad ; \qquad f_{11} \;\equiv\; cc_{12}^{D} \;-\; \frac{3}{4}\varepsilon\varepsilon_{33}^{S}\,hh_{31} \tag{40}$$

Also, from Eqs. 31 and 32,

$$\tilde{U}^{OC} \;=\; \tilde{U}^{SC} \;+\; \frac{hh_{31}}{2}\left(S_1 \;+\; S_2\right)\left(D_3^{SC} \;-\; D_3^{OC}\right) \tag{41}$$

where, from Eqs. 28 and 29,

$$D_3^{OC} \;-\; D_3^{SC} \;=\; \frac{\varepsilon\varepsilon_{33}^{S}\,hh_{31}\,t}{2}\left[\frac{1}{A}\int_A \left(w'' \;+\; w'/r\right)dA\right] \tag{42}$$

so,

$$\int_0^t \tilde{U}^{OC}\,dz \;=\; \int_0^t \tilde{U}^{OC}\,dz \;-\; \frac{\varepsilon\varepsilon_{33}^{S}\,hh_{31}^{2}\,t^{3}}{8}\,\frac{\left(w'' \;+\; w'/r\right)}{A}\int_A \left(w'' \;+\; w'/r\right)dA \tag{43}$$

After integrating over the thickness, integrate over the cross-sectional area to find the total energy density. Because only axisymmetric deflections are being considered, the integration over angle yields a factor of 2π; consequently,

$$U^{SC} \;=\; \frac{\pi\,t^{3}}{3}\left\{f_{11}\int\left(\left(w''\right)^{2} \;+\; \left(w'/r\right)^{2}\right)r\,dr \;+\; f_{12}\,2\int w''\,w'\,dr\right\} \tag{44}$$

and

$$U^{OC} \;=\; U^{SC} \;+\; \frac{\pi^{2}\,t^{3}\,\varepsilon\varepsilon_{33}^{S}\,hh_{31}^{2}}{2A}\left\{\int\left(w'' \;+\; w'/r\right)r\,dr\right\}^{2} \tag{45}$$

It is useful to re-write the integrals over r in terms of non-dimensional variables so that the integrations become functions only of the shape of the deflection; actual dimensions can be brought out into the leading factors. To do this, define

$$\xi \;\equiv\; w/w_0 \qquad ; \qquad \eta \;\equiv\; r/a \tag{46}$$

where w_0 is the maximum deflection and a is the disk radius. With these replacements, the short-circuit energy becomes

$$U^{SC} \;=\; \frac{\pi\,t^{3}\,w_0^{2}}{3a^{2}}\left\{f_{11}\int\left(\left(\xi''\right)^{2} \;+\; \left(\xi'/\eta\right)^{2}\right)\eta\,d\eta \;+\; f_{12}\,2\int \xi''\,\xi'\,d\eta\right\} \tag{47}$$

with the open-circuit energy similarly changed.

To further simplify the notation, it is convenient to introduce several "shape" integrals – that is, integrals that depend only on the shape of the deflection function:

$$I_{p0} = \int \left[\left(\frac{\partial^2 \xi}{\partial \eta^2} \right)^2 + \left(\frac{1}{\eta} \frac{\partial \xi}{\partial \eta} \right)^2 \right] \eta \, d\eta \tag{48}$$

$$I_{p1} = 2 \int \frac{\partial^2 \xi}{\partial \eta^2} \frac{\partial \xi}{\partial \eta} \, d\eta \tag{49}$$

$$I_Q = \int \left(\frac{\partial^2 \xi}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial \xi}{\partial \eta} \right) \eta \, d\eta \tag{50}$$

$$I_A = 2 \int \eta \, d\eta \tag{51}$$

The integrals I_{p0} and I_{p1} are two terms in the expressions for potential (strain) energy; I_Q is associated with charge; and I_A is defined so that

$$A = \pi a^2 I_A \tag{52}$$

If the electrode were continuous from center to outer edge, then I_A would be one. For a center-supported disk with a non-zero center-post radius or for an electrode that did not cover the entire disk, I_A would be less than one.

With these shape integrals,

$$V^{OC} = w_0 \, h h_{31} \frac{t^2}{a^2} \frac{I_Q}{I_A} \tag{53}$$

$$U^{SC} = \frac{\pi t^3}{3} \frac{w_0^2}{a^2} \left\{ f_{11} I_{p0} + f_{12} I_{p1} \right\} \tag{54}$$

$$U^{OC} = \frac{\pi t^3}{3} \frac{w_0^2}{a^2} \left\{ f_{11} I_{p0} + f_{12} I_{p1} + \frac{3}{2} \varepsilon \varepsilon_{33}^S h h_{31}^2 \frac{I_Q^2}{I_A} \right\} \tag{55}$$

G. Integrate D_3^{SC} (Eq. 36) over the electrode area to obtain the short-circuit charge, q^{SC}:

$$q^{SC} = -\pi \varepsilon \varepsilon_{33}^S h h_{31} t \int \left(w'' + w'/r \right) r \, dr = -\pi w_0 \varepsilon \varepsilon_{33}^S h h_{31} t \, I_Q \tag{56}$$

Define a blocked capacitance (that is, the capacitance with zero strain):

$$C_B = \frac{\varepsilon\varepsilon_{33}^S A}{t} = \frac{\varepsilon\varepsilon_{33}^S \pi a^2}{t} I_A \tag{57}$$

From Eqs. 53 and 56,

$$\frac{1}{2}\frac{\left(q^{SC}\right)^2}{C_B} = \frac{1}{2} C_B \left(V^{OC}\right)^2 = \frac{\pi t^3}{2}\frac{w_0^2}{a^2}\varepsilon\varepsilon_{33}^S h h_{31}^2 \frac{I_Q^2}{I_A} \tag{58}$$

and from Eqs. 54 and 55,

$$U^{OC} = U^{SC} + \frac{1}{2} C_B \left(V^{OC}\right)^2 \tag{59}$$

which shows explicitly the relationship between the pure electrical energy storage and the two potential energy terms.

To complete the construction of the equivalent circuit:

A. Integrate the deflection, w, over the face area and divide by the area to obtain the average displacement. Multiply this result by ω to obtain the average face velocity.

$$v_{avg} = \frac{2\pi\omega}{A}\int w\, r\, dr = \frac{\pi a^2}{A}\omega w_0\, 2\int \xi\,\eta\, d\eta \tag{60}$$

$$v_{avg} = \omega w_0 \frac{I_v}{I_A} \tag{61}$$

where another shape integral has been defined:

$$I_v = 2\int \xi\,\eta\, d\eta \tag{62}$$

B. Integrate the kinetic energy density over the ceramic volume to obtain the total kinetic energy, KE, in the ceramic.

$$\widetilde{KE} = \frac{1}{2}\rho v^2 = \frac{1}{2}\rho\omega^2 w^2 \tag{63}$$

$$KE = \frac{1}{2}\rho\omega^2 \int_A \int_0^t w^2\, dz\, dA = \pi \rho t \omega^2 \int w^2\, r\, dr \tag{64}$$

$$KE = \rho \pi a^2 t\, \omega^2 w_0^2 \int \xi^2 \eta\, d\eta = \rho \pi a^2 t\, \omega^2 w_0^2\, I_{KE} \tag{65}$$

where

$$I_{KE} = \int \xi^2 \eta\, d\eta \tag{66}$$

C. Write the blocked dielectric energy density and integrate to find the total blocked dielectric energy. The energy density is

$$\tilde{U}^{BD} = \frac{1}{2}(D_3 E_3)^{S=0} \tag{67}$$

From Eq. 10 with S_1 and S_2 equal to zero:

$$E_3 = \frac{1}{\varepsilon\varepsilon_{33}^S} D_3 \tag{68}$$

From Eq. 25 with S_1 and S_2 equal to zero:

$$D_3 = \frac{\varepsilon\varepsilon_{33}^S}{t} V \tag{69}$$

so

$$\tilde{U}^{BD} = \frac{1}{2}\varepsilon\varepsilon_{33}^S \left(\frac{V}{t}\right)^2 \tag{70}$$

This expression is independent of z and r so the total energy is simply

$$U^{BD} = \frac{1}{2}\varepsilon\varepsilon_{33}^S \left(\frac{V}{t}\right)^2 t\, A = \frac{1}{2}\frac{\varepsilon\varepsilon_{33}^S \pi a^2 I_A}{t} V^2 \tag{71}$$

which, by Eq. 57, is

$$U^{BD} = \frac{1}{2} C_B V^2 \tag{72}$$

D. Determine the equivalent circuit parameters. The mechanical mass is given by Eq. 18:

$$m = \frac{\partial^2 KE}{\partial v_{avg}^2} = \frac{I_A^2}{I_v^2} \frac{\partial^2 KE}{\partial (\omega w_0)^2} \tag{73}$$

With Eq. 65,

$$m = 2 \rho \pi a^2 t \frac{I_A^2 I_{KE}}{I_v^2} \tag{74}$$

The mechanical stiffness is given by Eq. 19:

$$k = \frac{\partial^2 U^{SC}}{\partial w_{avg}^2} = \frac{I_A^2}{I_v^2} \frac{\partial^2 U^{SC}}{\partial w_0^2} \tag{75}$$

With Eq. 54,

$$k = \frac{2}{3} \frac{\pi t^3}{a^2} \left(f_{11} I_{P0} + f_{12} I_{P1} \right) \frac{I_A^2}{I_v^2} \tag{76}$$

As an aside,

$$U^{SC} = \frac{1}{2} k w_{avg}^2 \tag{77}$$

The blocked capacitance has already been determined through Eq. 57. Alternatively, it is the second partial derivative of U^{BD} with respect to voltage. (Compare Eq. 71 to Eq. 57.)

The electromechanical conversion factor is the ratio of electrical current under short-circuit conditions to the average face velocity. This is identical to the ratio of electrical charge to average face displacement:

$$\phi = -\frac{\partial q^{SC}}{\partial w_{avg}} = \frac{I_A}{I_v} \frac{\partial q^{SC}}{\partial w_0} \tag{78}$$

With Eq. 56,

$$\phi = \pi \varepsilon \varepsilon_{33}^{S} h h_{31} t \frac{I_Q I_A}{I_v} \tag{79}$$

This completes determination of the basic set of equivalent-circuit parameters. The dielectric loss can be included via the loss tangent and C_B. There is, at present, no determination of mechanical loss. The equivalent circuit[9] is drawn in Fig. 5 using an explicit electrical-to-mechanical transformer and in Fig. 6 by transforming all elements to the electrical side.

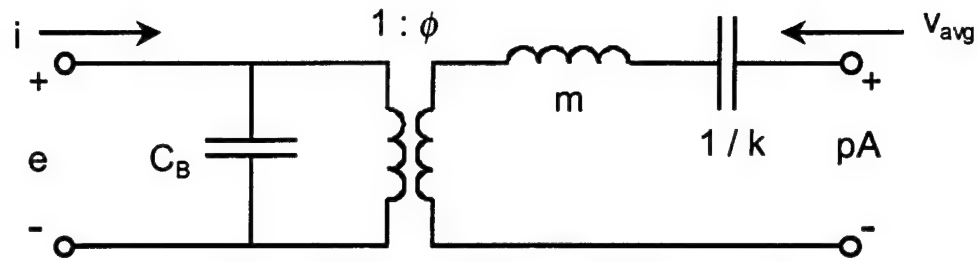

Fig. 5. Electromechanical equivalent circuit for flexural disk transducer with mechanical quantities to the right of the transformer and electrical quantities to the left of the transformer.

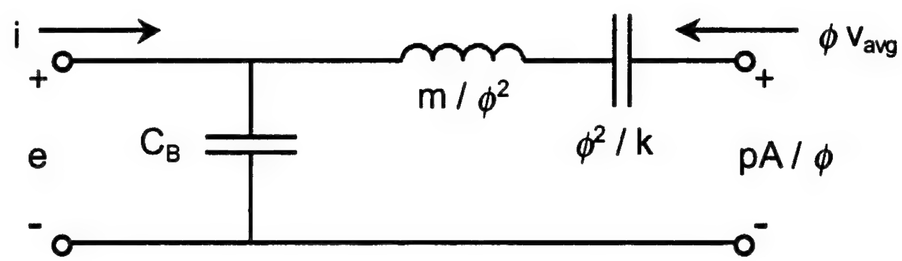

Fig. 6. Equivalent circuit with all quantities expressed in equivalent electrical terms.

E. Calculate the electromechanical coupling factor:

$$\kappa^2 = \frac{U^{OC} - U^{SC}}{U^{OC}} = \frac{\frac{1}{2} C_B V^2}{\frac{1}{2} k w_{avg}^2 + \frac{1}{2} C_B V^2} \tag{80}$$

(See Eqs. 59 and 77.) With Eqs. 54 and 55,

$$\kappa^2 = \frac{\frac{3}{2}\,\varepsilon\varepsilon_{33}^S\, hh_{31}^2\,\frac{I_Q^2}{I_A}}{f_{11}\, I_{p0} + f_{12}\, I_{p1} + \frac{3}{2}\,\varepsilon_{33}^S\, hh_{31}^2\,\frac{I_Q^2}{I_A}} \tag{81}$$

F. Calculate the resonance frequency:

$$\omega_{air}^2 = k\,/\,m \tag{82}$$

In this version, radiation loading is considered by including the approximate induced water mass. The water mass is calculated from the mass appropriate to a uniform, baffled piston of the same effective area as the actual piston. This radiation mass is

$$m_{rad} = \rho_{water}\,\pi\, a^2 \left(\frac{8\,a}{3\,\pi}\right) I_A \tag{83}$$

The water-loaded resonance frequency is then

$$\omega_{water}^2 = \frac{k}{m + m_{rad}} \tag{84}$$

The infinite baffle condition was chosen since it is a reasonable approximation in two common situations. The obvious case is if the transducer is mounted in a larger, nearly flat structure. Less obvious is the case in which the transducer comprises two flexural disk elements mounted back-to-back and driven to expand and contract in phase. In this case, the pair of sources can be replaced by a single source in an infinite baffle since the pair of sources represents the image-source solution to the baffled-source problem. Even if the element is not baffled, the error introduced is relatively small.

Treatment of an arbitrary neutral plane and incorporation of the dynamics associated with the substrate are straightforward modifications of the above analysis. The details are included in Appendix C, Appendix D, and Appendix E.

A preliminary treatment of stress in the transducer element is outlined in Appendix F and the results of the analysis in this report are compared with the results given in Woollett's report[1] in Appendix G. Appendix H contains the MathCad worksheet listings and Appendix I contains the materials properties files.

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Appendix Listing

Apppendix A: Exact Deflection Functions

For thin-plate theory, the general solution for the transverse displacement, w, as a function of radius, r, and angle, θ, is

$$w(r,\theta) = \cos(n\theta)\left[A_{n1} J_n(kr) + A_{n2} Y_n(kr) + A_{n3} I_n(kr) + A_{n4} K_n(kr)\right] \tag{A1}$$

where

$$k^4 = \frac{12\left(1-\sigma^2\right)\rho\omega^2}{E\, t^2} \tag{A2}$$

and E is the material's Young's modulus and σ is the Poisson's ratio. J_n and Y_n are the ordinary Bessel functions of the first and second kind, while I_n and K_n are the modified Bessel functions. Here, only the axially symmetric solutions will be considered ($n = 0$):

$$w(r) = A_1 J_0(kr) + A_2 Y_0(kr) + A_3 I_0(kr) + A_4 K_0(kr) \tag{A3}$$

To organize the boundary conditions, a number of auxiliary definitions will be introduced. These include displacement functions,

$$D_1 = J_0 \quad ; \quad D_2 = Y_0 \quad ; \quad D_3 = I_0 \quad ; \quad D_4 = K_0 \tag{A4}$$

slope functions,

$$S_1 = -J_1 \quad ; \quad S_2 = -Y_1 \quad ; \quad S_3 = I_1 \quad ; \quad S_4 = -K_1 \tag{A5}$$

radial moment functions,

$$M_1 = \frac{1-\sigma}{kr} J_1 \quad ; \quad M_2 = \frac{1-\sigma}{kr} Y_1 \tag{A6}$$

$$M_3 = -\frac{1-\sigma}{kr} I_1 \quad ; \quad M_4 = \frac{1-\sigma}{kr} K_1 \tag{A7}$$

and radial (Kelvin-Kirchhoff) shear functions,

$$V_1 = -J_1 \quad ; \quad V_2 = -Y_1 \quad ; \quad V_3 = -I_1 \quad ; \quad V_4 = K_1 \tag{A8}$$

These functions are combined to form the appropriate boundary conditions. For example, the condition for zero displacement at $r = a$ would be written as follows:

$$\sum_{i=1}^{4} A_i \, D_i(ka) = 0 \tag{A9}$$

The frequency constant is found by searching for the zeroes of the determinant of the matrix formed by the boundary condition functional forms. For edge-supported plates, there will be two conditions on the edge and the matrix will be 2x2; for plates supported by an inner support post, there will be two conditions at the support and two conditions at the outer edge so the matrix will be 4x4. If a is the outer radius, then define a frequency constant, λ^2, equal to $(ka)^2$. For a clamped edge, the deflection (D) and the slope (S) will be zero; for a simply-supported edge, the deflection (D) and the radial moment (M) will be zero; for a free edge, the radial moment (M) and the shear (V) will be zero.

For example, the boundary-condition matrix for a circular plate, simply-supported at the edge is

$$F = \begin{bmatrix} D_1(\lambda) & D_3(\lambda) \\ M_1(\lambda) & M_3(\lambda) \end{bmatrix} \tag{A10}$$

The functions with subscripts 2 and 4 are associated with the Bessel functions, Y, and K, which are infinite for zero argument; consequently, these functions are discarded in the solution for plates with continuity at $r = 0$.

As another example, the boundary-condition matrix for a circular plate, centrally clamped at $r = b$ and free at the outer edge is

$$F = \begin{bmatrix} D_1(\alpha\lambda) & D_2(\alpha\lambda) & D_3(\alpha\lambda) & D_4(\alpha\lambda) \\ S_1(\alpha\lambda) & S_2(\alpha\lambda) & S_3(\alpha\lambda) & S_4(\alpha\lambda) \\ M_1(\lambda) & M_2(\lambda) & M_3(\lambda) & M_4(\lambda) \\ V_1(\lambda) & V_2(\lambda) & V_3(\lambda) & V_4(\lambda) \end{bmatrix} \tag{A11}$$

where $\alpha = b/a$. In matrix form, the actual boundary conditions would be written

$$F \cdot A = 0 \tag{A12}$$

where A would either be 2x1 or 4x1:

$$A_{edge} = \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} \quad ; \quad A_{center} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \tag{A13}$$

In any case, the frequency constants are the values of λ for which

$$|F| = 0 \tag{A14}$$

The frequency constants, λ_m, are independent of the absolute disk dimensions and the disk material except for a modest dependence on Poisson's ratio (through the M functions). There are an infinite number of axially symmetric resonances but, here, only the first is important so the search for the roots of Eq. A14 would be restricted to the smallest value of λ.

Once the frequency constant for the lowest mode is found, the deflection function can be determined. For the edge-supported cases, there are two arbitrary constants (A_1 and A_3 from Eq. A13). At resonance, the two equations resulting from the matrix multiplication in Eq. A12 are not independent – either represents the solution. One of those equations determines the ratio of A_1 and A_3. Normalizing the equation so that the maximum deflection is one completely determines the constants.

For the centrally supported case, there are four arbitrary constants. Determination of the constants proceeds as follows: (1) Set A_1 to one. (2) Add the first two equations that result from the matrix multiplication in Eq. A12 together. (3) Form the 3x3 system of equations from step 2 and the remaining two equations with the constants A_2-A_4 on one side. (4) Solve this inhomogeneous equation system for A_2-A_4. (5) Normalize the result so that the maximum deflection is one.

Appendix B. Rayleigh-Ritz Approximation

Exact solution for the deflection function in the thin-plate approximation is straightforward as long as the disk is uniform in composition and thickness. In practice, however, the ceramic in most flexural disk transducers does not completely cover the substrate and this can have important consequences especially with regard to inner-edge stresses and electric-field breakdown. To anticipate the need to treat flexural-disk structures in which the ceramic only partially covers the substrate, an approximate energy-based technique for finding the deflection function is introduced in this second release. This solution is one of the Rayleigh-Ritz techniques. In the polynomial form of the Rayleigh-Ritz technique, the deflection function is assumed to have the form of a polynomial (normally with degree between two and ten). The resonance frequency is computed from the strain energy and the kinetic energy and then the resonance is minimized with respect to the coefficients of the polynomial. In this manner, several solutions are produced. Each solution is an approximation for a mode; the lower the mode, the better the approximation. Generally, the fundamental mode is approximated quite closely while the highest modes are approximated poorly.

The starting point for the Rayleigh-Ritz technique is construction of a function that satisfies the boundary conditions and that has a number of adjustable parameters. Then both the potential energy and the kinetic energy are expressed in terms of this function. The resonance frequency is obtained by setting the kinetic energy equal to the potential energy and solving for frequency. Minimizing the frequency with respect to all of the adjustable parameters produces an approximate solution for both the resonance frequencies and the mode shapes. One of the simplest implementations for the Rayleigh-Ritz function is as a polynomial.

Application to transverse vibrations of a circular membrane

To illustrate the polynomial Rayleigh-Ritz solution, we will start with a simpler problem: axially symmetric transverse vibrations of a circular membrane. After this introductory illustration, the technique will be applied to flexural vibrations of a thin circular plate.

First, write the displacement of the membrane as a polynomial in the radial coordinate. Here, we will use normalized forms. The variable, ξ, is the displacement divided by the peak displacement (at the center of the membrane). The variable, η, is the radial coordinate divided by the radius of the membrane.

$$\xi(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \ldots$$

The boundary conditions for the membrane are: (1) displacement equal to zero at the circumference, and (2) slope equal to zero at the center:

$$\xi(1) = 0 \text{ and } \frac{d\xi}{d\eta}(0) = 0$$

(Actually, the condition on the slope at the center is unnecessary. The accuracy of higher-order modes is slightly better using both conditions.)

The slope is

$$\frac{d\xi}{d\eta} = a_1 + 2a_2\eta + 3a_3\eta^2 + \ldots$$

so we set

$$\left.\frac{d\xi}{d\eta}\right|_{\eta=0} = a_1 = 0$$

and

$$\xi(1) = a_0 + a_1 + a_2 + a_3 + \ldots = 0$$

$$\therefore a_0 = -a_2 - a_3 - a_4 - \ldots$$

Therefore, the normalized deflection can be written as follows:

$$\xi(\eta) = -a_2 + a_2\eta^2 - a_3 + a_3\eta^3 - a_4 + a_4\eta^4 - \ldots$$

or

$$= a_2\left(\eta^2 - 1\right) + a_3\left(\eta^3 - 1\right) + a_4\left(\eta^4 - 1\right) + \ldots$$

Notice that, in the second form, each term satisfies the boundary conditions.

For convenience, we change the indexing to 1 ... N and replace a_{m+1} by b_m:

$$\xi(\eta) = \sum_{m=1}^{N} b_m\left(\eta^{m+1} - 1\right)$$

$$\frac{d\xi}{d\eta} = \xi' = \sum_{m=1}^{N} (m+1)\, b_m\, \eta^m$$

The kinetic energy of the membrane is

$$KE = \frac{1}{2}\int (\dot{w})^2\, dm = \frac{\omega^2}{2}\int_0^{2\pi}\int_0^a w^2 \rho\, t\, r\, dr\, d\phi$$

or, in terms of the normalized displacement,

$$KE = \omega^2 \, w_{peak}^2 \, \rho \, \pi \, a^2 \, t \int_0^1 \xi^2 \, \eta \, d\eta$$

The potential energy is equal to the work against the membrane tensile stress, σ. For an element, dr, of the membrane, the unstretched length is dr while the stretched length is the square root of dr^2 plus dw^2. The elongation is the difference between these two lengths which, for small deflection, is approximately

$$\frac{1}{2}\left(\frac{dw}{dr}\right)^2 dr$$

The force against which this elongation takes place is the product of the tensile stress and the area of the edge of the annular element of the membrane so

$$d\,PE = 2\pi \, r \, t \, \sigma \, \frac{1}{2}\left(\frac{dw}{dr}\right)^2 dr$$

The total potential energy (at maximum deflection) is then

$$PE = w_{peak}^2 \, \pi \, \sigma \, t \int_0^1 \left(\frac{d\xi}{d\eta}\right)^2 \eta \, d\eta$$

If we define U and T as follows:

$$U \equiv \int_0^1 \left(\frac{d\xi}{d\eta}\right)^2 \eta \, d\eta \quad ; \quad T \equiv \int_0^1 \xi^2 \, \eta \, d\eta$$

we can equate the kinetic and potential energies and solve for the frequency:

$$\omega^2 = \frac{\sigma}{\rho \, a^2}\left[\frac{U}{T}\right]$$

We can also define a normalized resonance frequency for membrane:

$$\alpha^2 \equiv \frac{U}{T}$$

To find the minima with respect to b_m:

$$\frac{d\alpha^2}{db_m} = \frac{T(dU/db_m) - U(dT/db_m)}{T^2} = 0$$

Set the numerator (divided by T) equal to zero:

$$\frac{dU}{db_m} - \frac{U}{T}\frac{dT}{db_m} = 0$$

which is the same as

$$\frac{dU}{db_m} - \alpha^2 \frac{dT}{db_m} = 0$$

This has the form of a matrix eigenvalue problem. In matrix terms, if $\mathbf{B}$ is a column vector of the coefficients, b_m, then

$$T = \mathbf{B}^{tr}\,\hat{\mathbf{T}}\,\mathbf{B}$$

and

$$\frac{dT}{db_m} = [0 \quad 0 \quad \ldots \quad 1 \quad 0]\,\hat{\mathbf{T}}\,\mathbf{B} + \mathbf{B}^{tr}\,\hat{\mathbf{T}}\,[0 \quad 0 \quad \ldots \quad 1 \quad 0]^{tr}$$

where the row vectors are zero except for a one in the m^{th} position. The column vector of the derivatives is then

$$\left[\frac{dT}{db_m}\right] = \hat{\mathbf{T}}\,\mathbf{B} + \left(\mathbf{B}^{tr}\,\hat{\mathbf{T}}\right)^{tr} = \left(\hat{\mathbf{T}} + \hat{\mathbf{T}}^{tr}\right)\mathbf{B}$$

Therefore,

$$\mathbf{T} = \hat{\mathbf{T}} + \hat{\mathbf{T}}^{tr}$$

and similarly for $\mathbf{U}$.

Since,

$$\frac{dU}{db_m}, \frac{dT}{db_m} \rightarrow \sum_n b_n \times \text{functions of } n, m$$

there are N equations each involving the N values of b_m:

$$\overset{(N\times N)}{\mathbf{U}} \cdot \overset{(N\times 1)}{\mathbf{B}} - \alpha^2\,\mathbf{T}\cdot\mathbf{B} = 0$$

where

$$\mathbf{U},\mathbf{T} = N\times N \text{ square matrices}$$

and

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \end{bmatrix} \qquad N\times 1$$

The equations are homogeneous so solutions only exist for specific values (eigenvalues) of α^2:

$$\left(\mathbf{U} - \alpha^2\,\mathbf{T}\right)\cdot\mathbf{B} = 0$$

Each α^2 has an associated $\mathbf{B}$ (the eigenvector or column vector of b_m's).

So α^2 could be written as a diagonal matrix:

$$\left[\alpha^2\right] = \begin{bmatrix} \alpha_{11}^2 & & 0 \\ & \alpha_{22}^2 & \\ 0 & & \alpha_{NN}^2 \end{bmatrix}$$

Since,

$$\mathbf{U} - \alpha^2\,\mathbf{T} = 0$$
$$\alpha^2\,\mathbf{T} = \mathbf{U}$$

$$\left[\alpha^2\right] = \text{diagonalization of } \mathbf{T}^{-1}\cdot\mathbf{U}$$

If we calculate $\mathbf{T}^{-1}\cdot\mathbf{U}$, then a standard matrix eigenvalue solver can be used to find both the eigenvalues and the eigenvectors.

Continuing the example of the membrane in tension, **U** and **T** are found as follows. The integrand of the kinetic energy is

$$\begin{aligned}\xi^2\eta &= \sum_m \sum_n b_m b_n \left(\eta^{m+1} - 1\right)\left(\eta^{n+1} - 1\right)\eta \\ &= \sum_m \sum_n b_m b_n \left[\eta^{m+n+3} - \eta^{m+2} - \eta^{n+2} - \eta\right]\end{aligned}$$

Integrate to find T:

$$T = \int_0^1 \xi^2 \eta \, d\eta =$$

$$\sum_m \sum_n b_m b_n \left[\frac{1}{m+n+4} - \frac{1}{m+3} - \frac{1}{n+3} - \frac{1}{2} \right]$$

then differentiate with respect to b_m:

$$\frac{dT}{db_m} = 2 \sum_\eta b_n \left[\frac{1}{m+n+4} - \frac{1}{m+3} - \frac{1}{n+3} - \frac{1}{2} \right]$$

The factors in square brackets are the matrix elements of $\hat{\mathbf{T}}$. Notice that $\hat{\mathbf{T}}$ is symmetric hence, when added to its transpose merely produces the leading factor of two; therefore, the elements of $\mathbf{T}$ are two times the quantities in square brackets.

The integrand of the potential energy is

$$\left(\frac{d\xi}{d\eta} \right)^2 \eta = \sum \sum b_m b_n (m + 1)\,(n + 1)\, \eta^{m+n+1}$$

so integrate to find U:

$$U = \int_0^1 \left(\frac{d\xi}{d\eta} \right)^2 \eta \, d\eta = \sum \sum b_m b_n \frac{(m + 1)\,(n + 1)}{m + n + 2}$$

and differentiate,

$$\frac{dU}{db_m} = 2 \sum_n b_n \left[\frac{(m + 1)\,(n + 1)}{m + n + 2} \right]$$

where the factors in square brackets are the matrix elements of $\hat{\mathbf{U}}$.

Application to transverse vibrations of a center-supported circular disk

For the centrally supported annular disk, the boundary conditions to be satisfied are for the displacement and slope at the inner edge to be zero. In energy-minimization methods, it is not necessary to satisfy explicitly conditions on moment or shear (at hinged or free boundaries, for example)[B1]. The process of minimization automatically satisfies these conditions on derivatives higher than the first. *Note that in this section, b refers to the inner radius (the radius of the clamp); it is not a polynomial coefficient.*

The polynomial solution that satisfies the clamped inner-edge boundary conditions is

$$\xi = \sum a_m \left[\eta^{m+1} - (m+1)\, b^m\, \eta + m\, b^{m+1} \right]$$

The first and second derivatives are

$$\frac{d\xi}{d\eta} = \sum a_m (m+1) \left[\eta^m - b^m \right]$$

$$\frac{d^2\xi}{d\eta^2} = \sum a_m\, m\, (m+1)\, \eta^{m-1}$$

We have already treated the normalization constants in the first release report so we need only construct the mode shape using Rayleigh-Ritz. Once we have the normalized shape, we can use the shape integrals previously derived to obtain the equivalent-circuit parameters and resonance frequency. Consequently, to find the shape, we can ignore all leading factors and assume an effective modulus and Poisson's ratio for the composite disk.

Since

$$T_1 = \frac{E}{1-\nu^2} [S_1 + \nu\, S_2]$$

$$T_2 = \frac{E}{1-\nu^2} [S_2 + \nu\, S_1]$$

the strain energy density is

$$U_{ST} = \frac{1}{2} S_1\, T_1 + \frac{1}{2} S_2\, T_2$$

$$= \frac{1}{2} \frac{E}{1-\nu^2} \left[S_1^2 + S_2^2 + 2\nu\, S_1\, S_2 \right]$$

[B1] R. Weinstock, *Calculus of Variations*, Dover, NY, 1974, §7.7.

Although this form is useable, it is commonly rearranged as follows:

$$U_{ST} = \frac{1}{2}\frac{E}{1-\nu^2}\left[\left(S_1 + S_2\right)^2 - 2\left(1-\nu\right)S_1 S_2\right]$$

Substituting for the strains,

$$\begin{aligned} U_{ST} &= \frac{z^2}{2}\frac{E}{1-\nu^2}\left[\left(w'' + \frac{w'}{r}\right)^2 - 2\left(1-\nu\right)\frac{w'' w'}{r}\right] \\ &= \frac{z^2}{2}\frac{E}{1-\nu^2}\left[\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)^2 - 2\left(1-\nu\right)\frac{1}{r}\frac{d^2 w}{dr^2}\frac{dw}{dr}\right] \end{aligned}$$

The integration over the volume of the composite disk produces several leading factors that can be ignored. The integration over the radial coordinate is necessary to retain but all the shape information is contained in the normalized forms. Therefore, let

$$U = I_{sum} - I_{p1}$$

where

$$I_{sum} = \int_b^1 \frac{1}{\eta}\left[\frac{d}{d\eta}\left(\eta\frac{d\xi}{d\eta}\right)\right]^2 d\eta$$

and

$$I_{p1} = 2\int_b^1 \frac{d^2\xi}{d\eta^2}\frac{d\xi}{d\eta}\, d\eta$$

(The notation of this last integral is consistent with that in the first report. I_{sum} is equal to the sum of I_{p0} and I_{p1}.)

Each integral produces a matrix. The algebra is straightforward although, in some cases, tedious. To illustrate, consider the integrand of I_{sum}:

$$\begin{aligned} \frac{1}{\eta}\left[\frac{d}{d\eta}\left(\eta\frac{d\xi}{d\eta}\right)\right]^2 &= \sum\sum a_m\, a_n\left(m+1\right)\left(n+1\right)\bullet \\ &\left[\left(m+1\right)\left(n+1\right)\eta^{m+n-1} - b^m\left(n+1\right)\eta^{n-1} - b^n\left(m+1\right)\eta^{m-1} + \frac{b^{m+n}}{\eta}\right] \end{aligned}$$

The integration produces the double sum of the product of a_m, a_n, and the matrix elements of the desired matrix:

$$\mathbf{U}_{sum}(m,n) = (m+1)(n+1) \cdot$$
$$\left[\frac{(m+1)(n+1)}{m+n}\left(1-b^{m+n}\right) - \frac{n+1}{n}b^m\left(1-b^n\right) - \frac{m+1}{m}b^n\left(1-b^m\right) - b^{m+n}\ln b\right]$$

In general, to produce the required matrices, we can take the complete factor that multiplies the product of a_m and a_n in the double summation, divide it by two and add it to its transpose. If the factor is symmetric in m and n, though, this step is redundant.

The result for I_{pl},

$$I_{pl} = \sum\sum a_m\, a_n\, 2\,n\,(m+1)(n+1)\left[\frac{1-b^{m+n}}{m+n} - \frac{b^m\left(1-b^n\right)}{n}\right]$$

is not symmetric in m and n so we would generate one matrix from half of the factor shown, then add it to its transpose to find $\mathbf{U}_{pl}$. The complete matrix for potential energy for the disk is then

$$\mathbf{U} = \mathbf{U}_{sum} - (1-\nu)\mathbf{U}_{pl}$$

The matrix for kinetic energy is found in similar fashion:

$$\begin{aligned}\mathbf{T}(m,n) = {} & \frac{1-b^{m+n+4}}{m+n+4} - (n+1)\,b^n\,\frac{1-b^{m+4}}{m+4} \\ & + n\,b^{n+1}\,\frac{1-b^{m+3}}{m+3} - (m+1)\,b^m\,\frac{1-b^{n+4}}{n+4} \\ & + (m+1)(n+1)\,b^{m+n}\,\frac{1-b^4}{4} - (2mn+m+n)\,b^{m+n+1}\,\frac{1-b^3}{3} \\ & + m\,b^{m+1}\,\frac{1-b^{n+3}}{n+3} + m\,n\,b^{m+n+2}\,\frac{1-b^2}{2}\end{aligned}$$

Once **U** and **T** have been found, the matrix eigenvalue/eigenfunction solver is used to find the fundamental mode frequency and mode shape. Matrix eigenvalue solvers do not usually return the results in order from lowest to highest mode; the smallest eigenvalue corresponds to the fundamental mode. The MathCad *rsort* function is used so that both the eigenvalue matrix and the eigenvector matrix can be sorted at the same time.

Once the mode shape has been found (that is, the polynomial coefficients have been determined from the eigenvector values), the deflection can be normalized so that the maximum deflection is

one for consistency with the first release. Then the other shape integrals can be calculated. If the normalized polynomial coefficient matrix is **C**, then,

$$
\begin{aligned}
I_{ke} &= \int_b^1 \xi^2 \eta \, d\eta = \mathbf{C}^{tr}\,\mathbf{T}\,\mathbf{C} \\
I_{psum} &= \mathbf{C}^{tr}\,\mathbf{U}_{sum}\,\mathbf{C} \\
I_{p1} &= \mathbf{C}^{tr}\,\mathbf{U}_{p1}\,\mathbf{C} \\
I_{p0} &= I_{psum} - I_{p1}
\end{aligned}
$$

As before, the shape integral associated with the effective area is

$$I_A = 2\int_b^1 \eta \, d\eta = 1 - b^2$$

The shape integral associated with surface charge is

$$I_Q = \int_b^1 \left(\xi'' + \frac{\xi'}{\eta} \right) \eta \, d\eta = \mathbf{C}^{tr}\,\mathbf{Q}$$

where

$$\mathbf{Q}(m) = (m+1)\left(1 - b^m\right)$$

and the shape integral associated with velocity is

$$I_v = 2\int_b^1 \xi \eta \, d\eta = \mathbf{C}^{tr}\,\mathbf{V}$$

where

$$\mathbf{V}(m) = 2\left[\frac{1-b^{m+3}}{m+3} - (m+1)\, b^m \frac{1-b^3}{3} + m\, b^{m+1} \frac{1-b^2}{2} \right]$$

For calculation of the deflection corresponding to maximum strain (see Section III), the following definition is useful:

$$\mathbf{SM}(m) = m\,(m+1)\, b^{m-1}$$

so that

$$\xi''(b) = \mathbf{C}^{tr}\,\mathbf{SM}$$

The matrix forms permit a cleaner program since **U**, **T**, **Q**, and **V** can be precomputed. Then the shape integrals are simple matrix multiplications with the coefficient matrix, **C**.

Application to transverse vibrations of a edge-supported circular disk

For the edge-supported disk (with simply-supported outer edge), the necessary boundary conditions are that the deflection be zero at the outer edge and that the slope be zero at the center:

$$\xi(1) = 0 \quad ; \quad \xi'(0) = 0$$

The normalized deflection function and its derivatives are then

$$\begin{aligned}
\xi &= \sum b_m \left(\eta^{m+1} - 1\right) \\
\xi' &= \sum b_m (m + 1)\eta^{m} \\
\xi'' &= \sum b_m m (m + 1)\eta^{m-1}
\end{aligned}$$

The required matrices are

$$\mathbf{U}_{sum}(m,n) = \frac{(m+1)^2 (n+1)^2}{(m+n)}$$

$$\hat{\mathbf{U}}_{pl}(m,n) = \frac{n(m+1)(n+1)}{(m+n)}$$

$$\mathbf{U}_{pl} = \hat{\mathbf{U}}_{pl} + \hat{\mathbf{U}}_{pl}^{tr}$$

$$\mathbf{U} = \mathbf{U}_{sum} - (1-\nu)\mathbf{U}_{pl}$$

$$\mathbf{T}(m,n) = \frac{1}{m+n+4} - \frac{1}{m+3} - \frac{1}{n+3} - \frac{1}{2}$$

$$\mathbf{Q}(m) = m + 1$$

$$\mathbf{V}(m) = \frac{2}{m+3} - 1$$

and

$$\mathbf{SM}(m) = m(m+1)b^{m-1}$$

The formulation in this release is sufficiently general that the edge-supported structure can be modeled either with or without a substrate. (Set the thickness of the substrate to zero in the latter case.) The edge-supported structure is assumed to be two identical ceramic disks laminated to each other or on opposite sides of a substrate disk.

Appendix C. Arbitrary Location of Neutral Plane

While there is some logic in locating the neutral plane at the interface between the ceramic and the substrate, this specific choice limits the design excessively. A module was added to compute the location of the neutral plane for arbitrary thickness of ceramic and substrate (within the limits of the thin-plate approximation).

The neutral plane can be found by minimizing the strain energy in the two-layer system. If the z-coordinate is defined so that $z = 0$ is the location of the neutral plane, then the strain in the system has the following relationship to the deflection, w:

$$S_1 = S_{rr} = -z\frac{d^2 w}{dr^2} = -z\,w''$$

$$S_2 = S_{\theta\theta} = -\frac{z}{r}\frac{dw}{dr} = -\frac{z\,w'}{r}$$

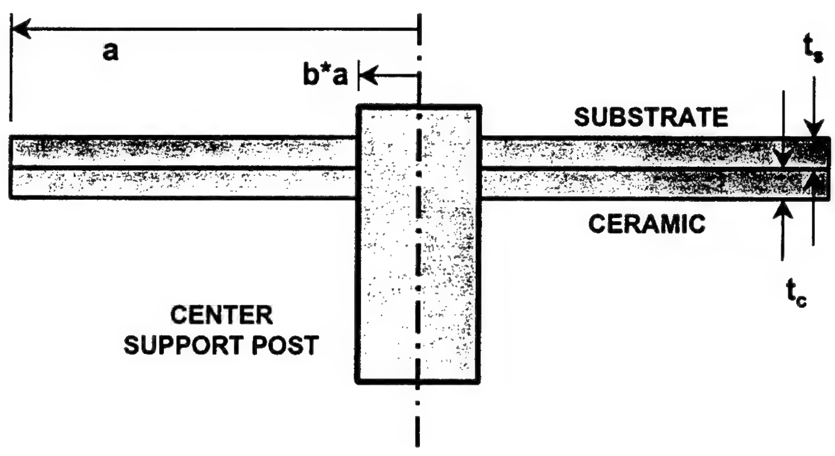


Fig. C1. Center-supported flexural disk structure. In this view, the water would be on the upper face and the lower face would contact a gas back-volume. For many practical transducers, the structure would be doubled so as to have two structures back-to-back. The edges are nominally free but would, in practice, be sealed with boots or bellows.

Let the ceramic be below the substrate and the ceramic thickness and substrate thickness be t_c and t_s respectively (see Fig. C1). If the neutral plane is at $z = 0$ and we assume that the neutral plane is in the substrate a distance z_0 from the interface (see Fig. C2), then the ceramic extends from $z = -t_c - z_0$ to $z = -z_0$, and the substrate extends from $z = -z_0$ to $z = t_s - z_0$.

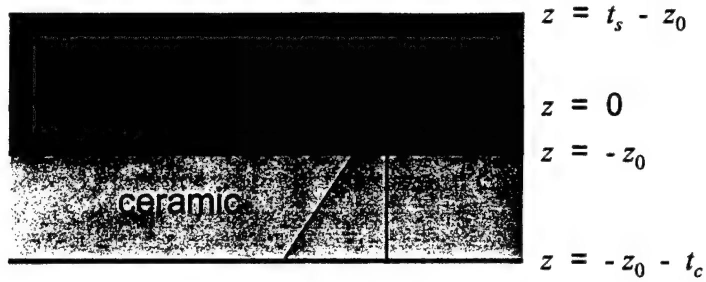


Fig. C2. Relationship of neutral plane (dash-dot line) to substrate/ceramic interface. The origin of the z-coordinate system is taken to be at the neutral plane.

The strain energy density is

$$\tilde{U}_{ST} = \frac{1}{2} S^{tr} T = \frac{1}{2} S_1 T_1 + \frac{1}{2} S_2 T_2$$

since all stresses other than T_1 and T_2 are zero.

The strain energy is the integral of the strain energy density over the entire structure. Integration over r and θ produce factors that do not depend on the location of the neutral plane, so these integrations do not need to be carried out explicitly. Since the faces of the ceramic are electroded, the electric flux density, D, does not depend on z so we will use the basic equation:

$$T = c^D S$$

which expands to

$$\begin{aligned} T_1 &= c_{11}^D S_1 + c_{12}^D S_2 + c_{13}^D S_3 \\ T_2 &= c_{12}^D S_1 + c_{11}^D S_2 + c_{13}^D S_3 \\ 0 &= c_{13}^D S_1 + c_{13}^D S_2 + c_{33}^D S_3 \end{aligned}$$

The last equation can be used to eliminate S_3 from the first two to produce:

$$T_1 = \left(c_{11} - \frac{c_{13}^2}{c_{33}} \right) S_1 + \left(c_{12} - \frac{c_{13}^2}{c_{33}} \right) S_2$$

For convenience, redefine the quantities in parentheses as the effective constants, cc^D:

$$T_1 = cc_{11}^D S_1 + cc_{12}^D S_2$$

Also,

$$T_2 = cc_{12}^D S_1 + cc_{11}^D S_2$$

(Note: The *cc* coefficients can also be obtained by the matrix relations discussed previously. While the forms are different –

$$cc_{11}^{D} = \frac{1/s_{11}^{D}}{1 - \left(s_{12}^{D}/s_{11}^{D}\right)^{2}}$$

$$cc_{12}^{D} = \frac{-s_{12}^{D}/\left(s_{11}^{D}\right)^{2}}{1 - \left(s_{12}^{D}/s_{11}^{D}\right)^{2}}$$

the values are identical.

For an isotropic material, the Young's modulus, E, is the reciprocal of s_{11} and the Poisson's ratio, ν, is negative the ratio of s_{12} to s_{11}. This produces the usual forms for thin plates:

$$T_1 = \frac{E}{1-\nu^2} S_1 + \frac{\nu E}{1-\nu^2} S_2$$

$$T_2 = \frac{\nu E}{1-\nu^2} S_1 + \frac{E}{1-\nu^2} S_2$$

It is useful to remember that, while T_3 is assumed to be zero, S_3 is, in general, not zero.)

The strain energy density is then

$$\tilde{U}_{ST} = \frac{cc_{11}}{2}\left(S_1^2 + S_2^2 + 2\nu S_1 S_2\right)$$

(We'll drop the superscript, D, on *cc* so that we can use the superscript to differentiate between the ceramic and the substrate.)

Actually, the strain energy also contains terms with the product of S and D. We will ignore these terms with respect to locating the neutral plane. Because the stiffness of a piezoelectric material depends on the electrical boundary conditions, the location of the neutral plane also must depend on these conditions. For the present, this is presumed to be a small effect; however, if the coupling factor is large, it may be necessary to modify this assumption. We can test the assumption by calculating the neutral-plane location using both cc^D and cc^E coefficients.

Each of the terms in the strain energy density equation is the product of two strains. Consequently, each term has as a factor z^2. This z^2 factor, cc_{11}, and ν are the only quantities that change with z (cc and ν because the material changes in the thickness direction). We will further assume that the Poisson's ratios of the two materials are not radically different. We will retain the dependence of cc on ν but ignore the change in ν as a factor of $S_1 S_2$.

Integration over r and θ produce factors that have no dependence on the location of the neutral plane; these factors can be ignored in the energy minimization. Integration over z produces a factor of z^3 times cc_{11}. Integration over the ceramic extends from $z = - t_c - z_0$ to $z = - z_0$ to produce

$$cc_{11}^{cer}\left[(-z_0)^3 - (-t_c - z_0)^3\right]$$

while integration over the substrate extends from $z = - z_0$ to $z = t_s - z_0$, producing

$$cc_{11}^{sub}\left[(t_s - z_0)^3 - (-z_0)^3\right]$$

If we add these two expressions together (for the total strain energy), differentiate with respect to z_0, set equal to zero, and solve for z_0, we obtain

$$z_0 = \frac{cc_{11}^{sub}\, t_s^2 - cc_{11}^{cer}\, t_c^2}{2\left[cc_{11}^{sub}\, t_s + cc_{11}^{cer}\, t_c\right]}$$

This gives the location of the neutral plane in terms of the material properties and thicknesses for the ceramic and substrate layers. If z_0 is negative, then the neutral plane is in the ceramic and a hydrostatic load would put part of the ceramic in tension unless prestressed.

Appendix D. Accounting for the Substrate

In the first release, the neutral plane was assumed to be at the interface between the ceramic and the substrate and the substrate properties were taken to be identical to the ceramic properties. This meant that only the ceramic needed to be analyzed. In subsequent releases, arbitrary properties and thickness were permitted for the substrate. Consequently, the calculations in the ceramic were modified (since the neutral plane is no longer, in general, at the inside face of the ceramic and calculations in the substrate were added.

The primary difference in the mechanics of calculation is that the limits of integration in the z-direction are no longer zero and t_c. Integrations over thickness in the ceramic go from $-z_0 - t_c$ to $-z_0$ and integrations over thickness in the substrate go from $-z_0$ to $t_s - z_0$. Before, integrations of S_1 or S_2 over the ceramic produced a factor of $t_c / 2$; now the same integrations produce a factor of

$$\frac{t_c}{2}\left[1 + \frac{2z_0}{t_c}\right] = \frac{t_c}{2} a_{c1}$$

where we have defined a constant, a_{c1}, equal to the factor in square brackets. Similar integration over the substrate produces a factor of

$$\frac{t_s}{2}\left[1 + \frac{2z_0}{t_s}\right] = \frac{t_s}{2} a_{s1}$$

If the integrations are of the squares of strains, then a factor of

$$\frac{t_c}{2}\left[1 + \frac{3z_0}{t_c} + \frac{9z_0^2}{t_c^2}\right] = \frac{t_c}{2} a_{c2}$$

is produced over the ceramic and a factor of

$$\frac{t_s}{2}\left[1 - \frac{3z_0}{t_s} + \frac{9z_0^2}{t_s^2}\right] = \frac{t_s}{2} a_{s2}$$

is produced over the substrate. (Notice the minus sign in this factor.)

Several quantities are unaffected by these changes. The blocked electrical capacitance is still

$$C_B = \frac{\varepsilon\varepsilon_{33}^S \pi a^2}{t_c} I_A$$

the effective area is

$$A = \pi a^2 I_A$$

and the average face velocity is

$$v_{avg} = w_0 \, \omega \frac{I_v}{I_A}$$

The open-circuit voltage changes to

$$V^{OC} = -w_0 \, hh_{31} \frac{I_q}{I_A} \frac{t_c^2}{a^2} a_{c1}$$

The potential energy in the ceramic for short-circuit electrical conditions becomes

$$U_{cer}^{SC} = \frac{w_0}{a^2} \frac{\pi t_c^3}{3} a_{c2} \left[f_{11} I_{p0} + f_{12} I_{p1} \right]$$

where

$$f_{11} = cc_{11}^D - \frac{3}{4} \varepsilon\varepsilon_{33}^S \, hh_{31}^2 \left(\frac{a_{c1}}{a_{c2}} \right)^2 \quad ; \quad f_{12} = cc_{12}^D - \frac{3}{4} \varepsilon\varepsilon_{33}^S \, hh_{31}^2 \left(\frac{a_{c1}}{a_{c2}} \right)^2$$

and the potential energy in the ceramic for open-circuit electrical conditions is

$$U_{cer}^{OC} = U_{cer}^{SC} + \frac{1}{2} C_B \left(V^{OC} \right)^2$$

The potential energy in the substrate is

$$U_{sub} = \frac{w_0}{a^2} \frac{\pi t_s^3}{3} a_{s2} \left[c_{11}^{sub} I_{p0} + c_{12}^{sub} I_{p1} \right]$$

where

$$c_{11}^{sub} = \frac{E_{sub}}{1 - \nu_{sub}^2} \quad ; \quad c_{12}^{sub} = \nu_{sub} \, c_{11}^{sub}$$

The total charge under short-circuit conditions is

$$q^{SC} \;=\; w_0\, \varepsilon\varepsilon_{33}^{S}\, hh_{31}\, \pi\, t_c\, a_{c1}\, I_A$$

and the kinetic energy is

$$KE \;=\; w_0^2\, \pi\, a^2\, \omega^2\, I_{KE}\left[\rho_c\, t_c \;+\; \rho_s\, t_s\right]$$

The elements of the equivalent circuit are defined in the same way but have somewhat different values. The equivalent mass is

$$m \;=\; \frac{\partial^2 KE}{\partial v_{avg}^2} \;=\; 2\pi\, a^2 \frac{I_A^2\, I_{KE}}{I_v^2}\left[\rho_c\, t_c \;+\; \rho_s\, t_s\right]$$

The equivalent stiffness is

$$k \;=\; \frac{\partial^2 U^{SC}}{\partial w_{avg}^2} \;=\; \frac{2\pi}{3a^2}\frac{I_A^2}{I_v^2}\left\{ t_c^3\, a_{c2}\left[f_{11}\, I_{p0} \;+\; f_{12}\, I_{p1}\right] \;+\; t_s^3\, a_{s2}\left[c_{11}^{sub}\, I_{p0} \;+\; c_{12}^{sub}\, I_{p1}\right]\right\}$$

where the potential energy is the sum of the energy in the ceramic and the energy in the substrate. The transduction factor is

$$\phi \;=\; \frac{\partial q^{SC}}{\partial w_{avg}} \;=\; \pi\, \varepsilon\varepsilon_{33}^{S}\, hh_{31}\, t_c\, a_{c1}\, \frac{I_Q\, I_A}{I_v}$$

The coupling factor is most easily calculated using the following form:

$$\kappa^2 \;=\; \frac{U^{OC} \;-\; U^{SC}}{U^{OC}} \;=\; \frac{\frac{1}{2}C_B\left(V^{OC}\right)^2}{U_{cer}^{SC} \;+\; U_{sub} \;+\; \frac{1}{2}C_B\left(V^{OC}\right)^2}$$

To estimate an upper limit to the transmitting voltage response, we assume operation at resonance and assume that the mechanical loss in the transducer is much less than the radiation resistance. Under these assumptions, the average face velocity is

$$v_{avg} \;=\; \frac{\phi\, V_{in}}{R_{rad}}$$

where V_{in} is the voltage applied to the electrical terminals and R_{rad} is the radiation resistance. The volume velocity can be computed from the effective area and the average face velocity.

Then, using the simple-source radiation expression, the ratio of pressure at one meter to applied voltage is

$$\frac{p}{V_{in}} = \rho_w \, f \, \pi \, a^2 \, I_A \, \frac{\phi}{R_{rad}}$$

In writing this expression, we have assumed that the transducer consists of two flexural-disk elements.

Appendix E. Treatment of the Trilaminar Structure

Analysis of a trilaminar flexural-disk transducer (see Fig. E1) is a straightforward extension of the two-layer transducer with arbitrary neutral plane. In the trilaminar transducer, the two ceramic layers are assumed to have identical properties and dimensions so that the neutral plane is midway through the substrate. If we take the origin of the *z*-coordinate system at the neutral plane, then integrations through thickness of the ceramic go from $z = - t_c - t_s/2$ to $z = - t_s/2$ and integrations through the thickness of the substrate go from $z = - t_s/2$ to $z = 0$. The results are then doubled to account for the symmetry of the structure about the neutral plane.

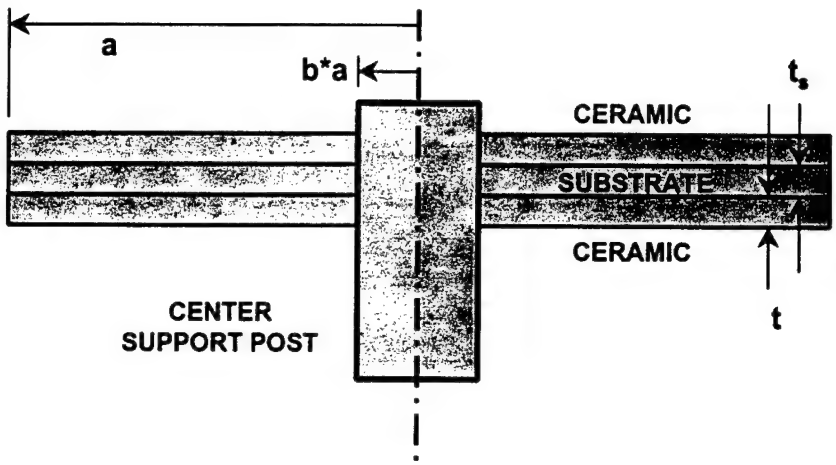


Fig. E1. Trilaminar center-supported flexural disk structure. The two ceramic layers are assumed identical. The neutral plane is in the middle of the substrate.

If we define the substrate thickness variable, t_s to be half the actual substrate thickness and let z_0 be equal to - t_s, then we can use the integration-limit constants derived for the two-layer, arbitrary neutral-plane formulation. We need only keep in mind that we are analyzing only half of the transducer structure.

In equivalent-circuit modules, the following changes are made. The equivalent mass and stiffness are both doubled. The coupling factor remains the same. The changes in other factors depend on the electrical connection of the two ceramic layers. If the ceramic layers are connected (electrically) in parallel, then the short-circuit charge, the blocked capacitance, and the transduction factor are all doubled, while the open-circuit voltage remains the same. If the ceramic layers are connected in series, then the short-circuit charge and the transduction factor remain the same while the open-circuit voltage doubles and the blocked capacitance is reduce by a factor of two. For this release version, the parallel electrical connection is assumed (but conversion of the parameters for series connection is trivial).

Appendix F. Stress Calculation and Interpretation

An important part of the design process is determination of the maximum source level achievable from the transducer. One of the critical limits is stress in the ceramic. If the desired source level is prescribed, then, the required volume velocity, Q, can be determined:

$$Q = \frac{2\,p}{\rho\, f}$$

where p is the desired acoustic pressure at one meter, ρ is the density of the fluid, and f is the frequency.

Using the dimensions of the transducer and the volume factor from the shape integral, I_v, the peak deflection that would produce that volume velocity is,

$$w_{peak} = \frac{Q}{2\pi\, f\, \pi\, a^2\, I_v}$$

Then, having the peak deflection and the deflection shape, the stresses can be calculated. In this second release, only the normalized stresses are calculated. The actual stresses can be calculated manually as outlined above.

The radial and tangential strains are determined by the deflection function, w,

$$S_1 = -z\frac{d^2 w}{dr^2} \quad ; \quad S_2 = -\frac{z}{r}\frac{dw}{dr}$$

The stresses can be determined from the following set of equations:

$$\begin{aligned} T_1 &= cc_{11}^D\, S_1 + cc_{12}^D\, S_2 - hh_{31}\, D_3 \\ T_2 &= cc_{12}^D\, S_1 + cc_{11}^D\, S_2 - hh_{31}\, D_3 \\ E_3 &= -hh_{31}\,(S_1 + S_2) + \beta\beta_{33}^S\, D_3 \end{aligned}$$

From the orientation of the electrodes, D_3 is constant in the 3-direction so we can integrate the third equation over the thickness of the ceramic to find D_3 in terms of the strain and the voltage on the electrodes.

The electrode voltage is calculated by integrating the electric field from one electrode to the other. In most published analyses of piezoelectric transducers, the sign is handled incorrectly. The voltage (electrical potential) is *minus* the integral of electric field. (The electric field is *minus* the gradient of the potential.) For the purposes of analysis of transducer performance, this error is generally inconsequential; it can be "rectified" simply by reversing the polarity of the voltage terminals in the equivalent circuit. However, when analyzing stress and the coupling

between mechanical and electrical properties, the sign is important. The first report did not address this sign error. To avoid confusion in this second report, the terminal voltage will be defined so that a positive voltage applied to the electrodes produces an electric field in the positive 3-direction. With the coordinate definitions used so far, this means that the "ground" terminal is the electrode between the substrate and the ceramic and the "positive" terminal is the other electrode. If the terminal voltage is V_t, then

$$-V_t = -\int E_3 \, dz$$

where the integration is taken from the lower electrode to the upper electrode. Therefore,

$$-V_t = hh_{31} \int_{-t_c - z_0}^{-z_0} (S_1 + S_2)\, dz - \beta\beta_{33}^S D_3 t_c$$

Since

$$S_1 + S_2 = -z\left(w'' + \frac{w'}{r}\right)$$

the integration is straightforward:

$$-V_t = \frac{1}{2} hh_{31}\left(w'' + \frac{w'}{r}\right)\left(t_c^2 + 2 t_c z_0\right) - \beta\beta_{33}^S D_3 t_c$$

or

$$D_3 = \frac{hh_{31}\left(t_c + 2 z_0\right)}{2\beta\beta_{33}^S}\left(w'' + \frac{w'}{r}\right) + \frac{V_t}{\beta\beta_{33}^S t_c}$$

The maximum stress in the ceramic is at the outer (lower) face where $z = -t_c - z_0$:

$$T_1^{\max} = \left(t_c + z_0\right)\left(cc_{11}^D w'' + \frac{1}{r} cc_{12}^D w'\right) - hh_{31} D_3$$

$$T_2^{\max} = \left(t_c + z_0\right)\left(cc_{12}^D w'' + \frac{1}{r} cc_{11}^D w'\right) - hh_{31} D_3$$

Inclusion of the D_3 term in the stress calculation will be deferred to the next release so the stresses calculated in the second release should be considered cautiously.

As an interim solution, the limiting performance can be developed in terms of maximum normal strain. This is not the preferred criterion for failure of brittle material (maximum tensile stress is the preferred limit) but it will serve as a rough approximation in this second release. For cases of

practical interest, the maximum normal strain is equal to the radial strain at the inner edge (and outer surface) of the disk:

$$S_1^{\max} = S_1\left(z = -t_c - z_0\right) = \left(t_c + z_0\right)\frac{d^2 w}{dr^2}$$

It is simpler to normalize the deflection function, w, in the following manner,

$$\xi \equiv w / w_0 \quad ; \quad \eta \equiv r / a$$

where w_0 is the maximum value of the deflection (at the edge of the center-supported disk or at the center of the edge-supported disk) and a is the disk radius. The maximum strain can then be written

$$S_1^{\max} = \left(t_c + z_0\right)\frac{w_0}{a^2}\xi''(b)$$

where the primes denote differentiation with respect to η. If we specify a maximum strain value, this expression permits us to find the corresponding peak deflection:

$$w_0^{\max} = \frac{a^2 S_1^{\max}}{\left(t_c + z_0\right)\xi''(b)}$$

(If we ignore the stress component related to D_3, we could find the peak deflection from a maximum stress criterion. The first derivative of ξ is zero at $\eta = b$, so

$$w_0^{\max} = \frac{a^2 T_1^{\max}}{c c_{11}^{D}\left(t_c + z_0\right)\xi''(b)}$$

for maximum normal stress.)

Having the peak deflection, the radiated pressure can be found from the simple-source approximation:

$$v_{avg}^{\max} = \omega\, w_{avg}^{\max} = \omega\, w_0^{\max}\frac{I_v}{I_A}$$

where the shape integrals, I_v and I_A, are defined in the next section;

$$Q^{\max} = v_{avg}^{\max} A = \pi a^2 I_A v_{avg}^{\max}$$

and

$$p(1\,m) = 2\frac{\rho f}{2}Q^{\max}$$

for the pressure at one meter (assuming the transducer contains two flexural disk elements).

Appendix G: Relationship of Solutions to Reference 1

Some adjustment is necessary to compare results with Woollett[G1]. In that reference, the deflection functions for the edge-supported cases are normalized so that the maximum deflection is one; however, the deflection function for the center-supported case is not.

The normalized deflection function, ξ, used here always has a maximum value of one. The deflection function amplitude used in Ref. 1 is a_i, where a_i is one for the edge-supported cases and a_i is 0.369 for the center-supported case.

Several of the constants used in Ref. 1 are similar to the shape integrals used here:

$$\Lambda^D = \frac{a_i^2}{8}\left[I_{P0} + \frac{cc_{12}^D}{cc_{11}^D} I_{P1} \right]$$

$$\Lambda^{\square} = \frac{a_i^2}{8}\left[I_{P0} + \frac{cc_{12}^D}{cc_{11}^D} I_{P1} - \frac{3}{4}\frac{\varepsilon\varepsilon_{33}^S \, hh_{31}^2}{cc_{11}^D}\left(I_{P0} + I_{P1}\right) \right]$$

$$K = 2 a_i^2 I_{KE}$$

$$H = a_i^2 \frac{I_v^2}{I_A^2}$$

Using these associations, the results from Ref. 1 can be compared to the present results by replacing the passive substrate used here with another ceramic layer having the same properties and dimensions as the first one. The comparisons are not exact because the basic deflection functions are different.

[G1] R. S. Woollett, "Theory of the piezoelectric flexural disk transducer with applications to underwater sound," USL Research Report No. 490, U.S. Navy Underwater Sound Laboratory, December 5, 1960.

Appendix H: MathCad Code Listing

The following modules are listed in this appendix. Please read the current FlexIntro file supplied with the MathCad files for updated formats and program modules.

FlexIntro3_0.mcd

CC2L_3_0config.mcd
CC2L_3_0deflect.mcd
CC2L_3_0equivckt.mcd

CC3L_3_0config.mcd
CC3L_3_0deflect.mcd
CC3L_3_0equivckt.mcd

ES2L_3_0config.mcd
ES2L_3_0deflect.mcd
ES2L_3_0equivckt.mcd

ES3L_3_0config.mcd
ES3L_3_0deflect.mcd
ES3L_3_0equivckt.mcd

Performance3_0.mcd

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Introduction

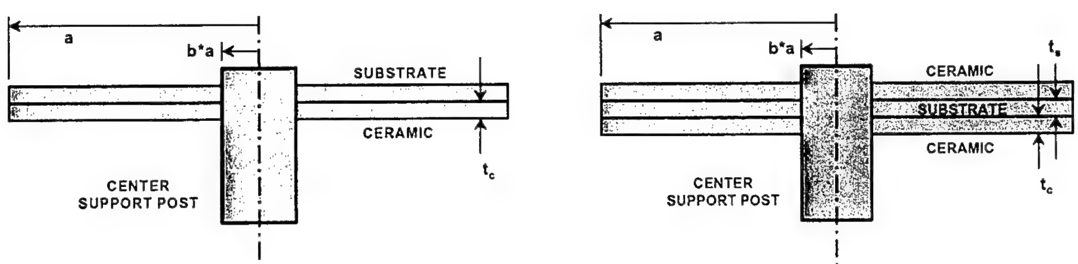
This model presents an analytical solution for the flexural-disk transducer in a manner similar to that used by Woollett (Theory of the Piezoelectric Flexural Disk Transducer with Applications to Underwater Sound, USL Report 490, December 5, 1960). The primary focus is on transmitting transducers with center-supported disks but edge-supported disks and receiving performance will also be treated.

The three most apparent differences between this work and Woollett's report are: (1) higher-order deflection functions are used to describe operation in the vicinity of resonance, (2) a non-zero diameter center support is considered, and (3) arbitrary location of the neutral plane is permitted. Allowing a non-zero center support diameter is critical in characterizing the stress distribution. Arbitrary location of the neutral plane permits modeling of two-layer (ceramic/substrate) disks.

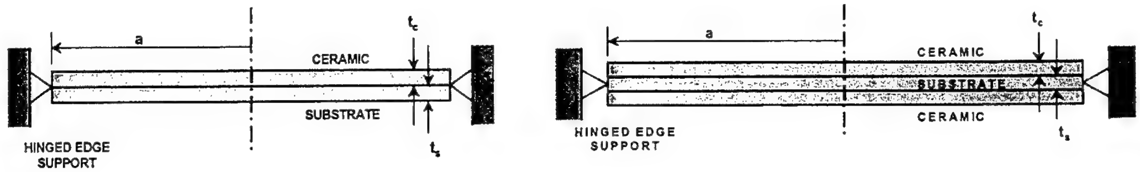
Users should be aware that this is a draft version rather than a fully validated production release.

Transducer Types

Four configurations of the flexural-disk transducer are analyzed. Two configurations are center-supported. The first center-supported configuration consists of a single ceramic layer and a substrate layer; the second consists of a substrate layer and two ceramic layers on either side of the substrate. These configurations are shown below:



Two configurations are edge-supported. The first edge-supported configuration consists of a single layer of ceramic and a substrate layer; the second consists of a substrate layer and two ceramic layers on either side of the substrate. In each case, the composite disk is simply supported at the edge:



Third Release

In this third release, the elements in the two-layer structures are considered to be two-layer laminations of ceramic and substrate with arbitrary properties and thicknesses. Consequently, the neutral plane is not restricted to being located at the ceramic-substrate interface. Some caution must be exercised now in that it is possible to put the neutral plane in the ceramic, which can cause failure under hydrostatic load if the structure is not prestressed. Also, in this second release, an approximate method for calculating the mode shape and shape integrals is used. Incorporation of this energy-based approximate method will eventually permit modeling of structures in which the ceramic does not completely cover the substrate. Version 3.0 calculates the equivalent mechanical resistance and the dielectric loss as equivalent-circuit elements.

There are three important additions in version 3.0. First, the three-layer edge-supported disk structure is treated (ES3Lxxx). Second, a feature has been added to the Performance Module that permits incorporation of a fluid-filled cavity behind the flexural disk. Finally, physical units have been incorporated in the Performance module. Since files do not preserve the units for quantities, the calculations in Performance were done previously without units (but assuming SI units). In this version, the units are now explicit (and SI).

As in the previous version, this release is presented as a series of modules. This architecture is described below and is convenient for exploring parameter variations.

Assumptions

A number of assumptions have been made in this version of the flexural disk model. Several of these assumptions will be relaxed as the model is developed; some may be retained in order to preserve the analytical nature of the model.

Thin-plate theory (plane stress; no shear)
Ceramic layer extends over entire diameter of disk structure

Simple assumed deflection functions are not used as there is no significant saving in computation nor is there any reduction in the analytical nature of the model by using "exact" mode functions or mode functions developed by Rayleigh-Ritz from higher order polynomials. When calculation of hydrostatic stresses are added to the model, the exact static deflection will be used for the same reasons.

Organization

The naming convention for the modules is as follows. The first part of the name indicates the transducer configuration. CC2L is the center-supported, two-layer configuration; CC3L is the center-supported, three-layer configuration; ES2L is the edge-supported, two-layer configuration; and ES3L is the edge-supported, three-layer configuration. For each transducer type, there are three modules and these are indicated by the second part of the module name. The first module is the configuration module (xxxx_config), the second (xxxx_deflect) is the module that finds the mode deflection function and the shape integrals, while the third module (xxxx_equivckt) computes the equivalent-circuit parameters and some of the important operating parameters.

Run one of the configuration modules (CC2L_config, CC3L_config, ES2L_config, or ES3L_config) first to define the transducer configuration, the piezoelectric material, and the substrate material. Many of the parameters subsequently calculated do not depend on specific dimensions. In those cases, dummy dimensions can be entered. The first release retained sufficient flexibility to calculate impractical limiting cases (such as the center-hole radius going to zero). Unrealistically small center radii should not be used in this version.

The configuration module also reads the piezoelectric parameters and substrate properties from the materials files and derives the necessary plane-stress forms of other parameters for subsequent calculation. Manufacturers' tabulated values MUST NOT BE USED for the *c* or *h* parameters or for the clamped dielectric permittivity, $\square^{S}$. Tabulated values are not appropriate for 2D plane-stress problems.

Then run one of the deflection-function modules (CC2L_deflect, CC3L_deflect, ES2L_deflect, or ES3L_deflect). This module finds the first mode eigenvalue and the corresponding deflection shape. Then the so-called shape integrals -- factors that depend only on the shape of the deflection curve -- are computed. The shape integrals are used in the calculation of equivalent-circuit parameters and other performance parameters.

Finally, run the parameter module (CC2L_equivckt, CC3L_equivckt, ES2L_equivckt, or ES3L_equivckt) to evaluate the equivalent-circuit parameters for the transducer configuration specified in the configuration module. Once the equivalent-circuit parameters have been stored, Performance can be run to compute admittance, TVR, TCR, and a few other operating properties. The inputs to set up the optional fluid-filled cavity are entered in Performance.

File Structure

Piezoelectric materials files (PZT4.prn, PZT5H.prn, PZT8.prn):

Each piezoelectric material file contains seven values from various manufacturer data sheets. The seven values are:

$s^{\square}_{12}$ $s^{\square}_{12}$ $\square^{T}_{33}/\square_{0}$ d_{31} □□□□□an□□ Qmech

Files for other materials can be generated easily. Create an ASCII file (use the extension .prn) with these seven values entered on a single line separated by spaces. The values are stored in the file as they are usually listed in properties tables: the compliances, s, in units of 10^{-12} m^2/N; the dielectric permittivity as relative dielectric permittivity; and the d coefficient in units of 10^{-12} m/V. These values are the same for 3D and 2D problems so they may be transcribed directly from manufacturer data sheets. The configuration module reads these values and then multiplies the compliances and the d coefficient by 10^{-12} and multiplies the relative dielectric permittivity by the permittivity of free space.

Substrate properties files (brass.prn, aluminum.prn, steel.prn):

The substrate properties files each contain three values: the density in kg/m^3, the elastic modulus in GPa, and the Poisson's ratio. The user can generate other substrate files as ASCII files (with .prn extension) as described above.

Configuration files (xxxx_config.prn):

The configuration files contain 15 values -- three dimensions and seven physical properties of the ceramic, three properties and the thickness of the substrate, and the location of the neutral plane:

$\square_c$ a b t_c c^{D}_{11} c^{D}_{12} h_{31} $\square^{S}_{33}$ tan□ Qmech ρ_{sub} E_{sub} ν_{sub} t_{sub} z_0

Units are not retained in the write operation so all values are converted to proper SI units before being written. User input values can be entered in other systems as long as they are entered with those units. If no units are specified, SI units will be assumed. The outer radius of the disk is a; the thickness is t_c; and the ratio of inner radius to outer radius is b. For the edge-supported disk, b is set to zero in the configuration module. The density and the loss tangent are transferred directly from the properties file. The stiffnesses, c^D, the h coefficient, and the clamped permittivity are derived based on the plane-stress assumption appropriate to the plate bending equations. These values will not compare to published values.

Shape files (xxxx_deflect.prn):

The shape files hold the values for the shape integrals; the peak deflection, wnmax, for unity strain; the size, N, of the coefficient matrices; and the polynomial coefficient matrix, CC:

I_{p0} I_{p1} I_q I_v I_{KE} I_a wnmax N CC

The shape integrals are functions only of the shape of the deflection function and so permit rather general conclusions to be drawn about different configurations without substituting specific dimensions or physical properties. Their value is not fully exploited by this initial draft. The first two, I_{p0} and I_{p1}, are related to elastic strain energy. The third, I_q, is related to the charge produced under strain. The fourth, I_v, relates the average face velocity to the peak face velocity. The fifth, I_{KE}, is related to the kinetic energy. And the sixth, I_a, is related to the active face area. The shape integrals and the configuration files are used to compute the equivalent-circuit parameters and various performance measures. The peak deflection for unity strain can be used to calculate performance limits (this is implemented in the third module for each configuration). N and CC can be used to construct the actual deflection function.

Circuit parameter files (xxxx_equivckt.prn):

The equivalent-circuit parameter files hold the following values:

mass stiffness □ C_{EB} κ^2 $\tan\delta$ Rmech r_rad m_rad Aeff

These parameters are the equivalent mass and stiffness at resonance; the electromechanical transduction factor, □; the blocked electrical capacitance, C_{EB}; the coupling factor, □2; the loss tangent, tan□, of the ceramic; and the equivalent mechanical resistance, R_{mech}, of the composite. The quantites, r_rad and M_rad are the radiation resistance and radiation mass for the parallel equivalent for radiation impedance. Aeff is the effective piston area.

Results

Mode deflection function
Basic energy quantities
Unloaded and loaded resonance frequency
Coupling factor
Equivalent-circuit parameters
Admittance (unloaded and loaded)
Transmitting voltage response (TVR)
Transmitting current response (TCR)
Low-frequency free-field voltage sensitivity (FFVS)
Calculation of maximum stress in ceramic and substrate

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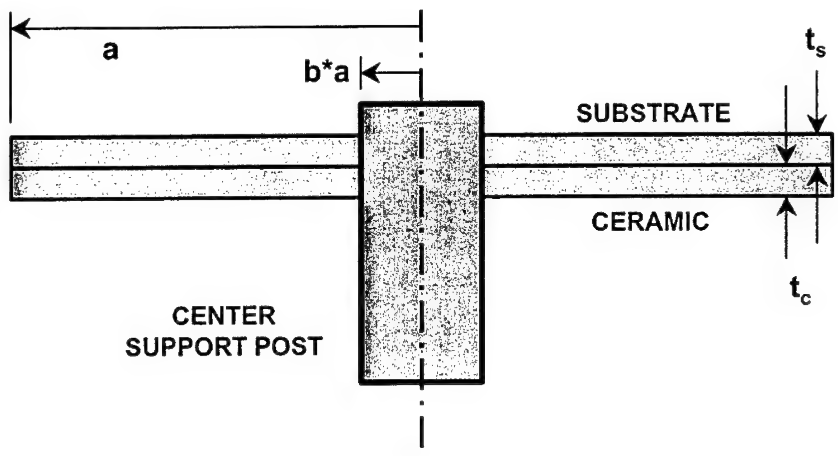
Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 1.CC2L_config: Configuration Definition - Center-Supported Disk

This worksheet is specific to the annular (center-supported) disk. First, the physical dimensions are specified and the piezoelectric properties file is read. Then, the two-dimensional properties are derived. Finally, the results are written to a configuration file.



I. SELECT MATERIALS [Highlighted values are user inputs]

Choose from PZT4, PZT5H, or PZT8 for file name in pzt READPRN.

pzt := READPRN(PZT4)

$se11 := pzt_{0,0} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $se12 := pzt_{0,1} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $\varepsilon t33 = pzt_{0,2} \cdot \varepsilon 0$

$d31 := pzt_{0,3} \cdot 10^{-12} \cdot \frac{m}{volt}$ $\rho := pzt_{0,4} \cdot \frac{kg}{m^3}$ $\tan\delta := pzt_{0,5}$ $Qmech = pzt_{0,6}$

Choose from steel, aluminum, or brass for file name in substrate READPRN.

substrate := READPRN(brass)

$\rho sub := substrate_{0,0} \cdot \frac{kg}{m^3}$ $Esub := substrate_{0,1} \cdot 10^9 \cdot Pa$ $vsub := substrate_{0,2}$

$\rho sub = 8.5 \cdot 10^3 \cdot kg \cdot m^{-3}$ $Esub = 1.04 \cdot 10^{11} \cdot kg \cdot m^{-1} \cdot sec^{-2}$ $vsub = 0.37$

II. SELECT DIMENSIONS

Enter dimensions of ceramic disk (a = outside radius; bi = inner radius; tc = thickness) and thickness (tsub) of substrate disk:

[Enter units for in, mm, cm, or m; if no units are entered, meters will be assumed]

$a := 7\cdot cm$ $\qquad bi := 1\cdot cm$ $\qquad tc := 2.5\cdot mm$ $\qquad tsub := 4\cdot mm$ $\qquad b := \frac{bi}{a}$

$a = 0.07\cdot m$ $\qquad b = 0.143$ $\qquad tc = 2.5\cdot 10^{-3}\cdot m$

III. DERIVE PROPERTIES FOR TWO-DIMENSIONAL ANALYSIS

$$sd11 := se11 - \frac{d31^2}{\varepsilon t33} \qquad sd12 := se12 - \frac{d31^2}{\varepsilon t33} \qquad -\frac{sd12}{sd11} = 0.485$$

$$cd11 := \frac{sd11}{sd11^2 - sd12^2} \qquad cd12 := \frac{-sd12}{sd11^2 - sd12^2} \qquad \frac{cd12}{cd11} = 0.485$$

$$\varepsilon s33 := \varepsilon t33 - \frac{2\cdot d31^2}{se11 + se12} \qquad h31 := \frac{d31}{\varepsilon t33\cdot(sd11 + sd12)}$$

IV. CALCULATE LOCATION OF NEUTRAL PLANE

Origin of z-coordinate at neutral plane; z0 gives distance from neutral plane to ceramic substrate interface:

$$fs := \frac{Esub}{1 - vsub^2} \qquad z0 := \frac{tsub^2\cdot fs - tc^2\cdot cd11}{2\cdot(tsub\cdot fs + tc\cdot cd11)} \qquad z0 = 7.605\cdot 10^{-4}\cdot m$$

WARNING: If z0 is negative, then the neutral plane is in ceramic and some part of the ceramic will be in tension under hydrostatic load unless prestressed.

V. WRITE CONFIGURATION FILE

PRN files cannot handle dimensions so all quantities are written (in SI) without units.

$$out_{0,0} := \frac{\rho}{\rho 0} \qquad out_{0,1} := \frac{a}{m0} \qquad out_{0,2} := b \qquad out_{0,3} := \frac{tc}{m0}$$

$$out_{0,4} := \frac{cd11}{c0} \qquad out_{0,5} := \frac{cd12}{c0} \qquad out_{0,6} := \frac{h31}{h0} \qquad out_{0,7} := \frac{\varepsilon s33}{e0}$$

$$out_{0,8} := \tan\delta \qquad out_{0,9} := Qmech \qquad out_{0,10} := \frac{\rho sub}{\rho 0} \qquad out_{0,11} := \frac{Esub}{c0}$$

$out_{0,12} := vsub$ $\qquad$ $out_{0,13} := \frac{tsub}{m0}$ $\qquad$ $out_{0,14} := \frac{z0}{m0}$

$WRITEPRN(CC2L\_config) := out$

The configuration file contains the following quantities in this order:

density of ceramic, ρ
outer radius of disk, a
ratio of inner radius to outer radius, b
thickness of ceramic, t
cd11 for ceramic
cd12 for ceramic
h31 for ceramic
εs33 for ceramic
loss tangent for ceramic, tanδ
mechanical Q of ceramic
density of substrate, ρsub
modulus of substrate, Esub
poisson's ratio of substrate, vsub
thickness of substrate, tsub
z location of neutral plane, z0, with respect to bottom of ceramic

Unit normalization quantities (predefined):

$\rho0 \equiv 1 \cdot \frac{kg}{m^3}$ $\qquad$ $m0 \equiv 1 \cdot m$ $\qquad$ $c0 \equiv 1 \cdot Pa$ $\qquad$ $h0 \equiv 1 \cdot \frac{volt}{m}$ $\qquad$ $e0 \equiv 1 \cdot \frac{farad}{m}$

$\varepsilon0 \equiv 8.854 \cdot 10^{-12} \cdot \frac{farad}{m}$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 2.CC2L_deflect: Center-Supported Disk -- Clamped Inside

This worksheet reads the configuration file, calculates the first axisymmetric mode deflection function, and then calculates all of the shape integrals. The shape integrals are written to a file for subsequent use. In this version, the polynomial Rayleigh-Ritz method is used to compute the mode deflection function. An eighth-order polynomial is used, which produces accurate results in all cases except unrealistically small inner radii. This module is expecting the configuration file from version 2 -- that is, the configuration with arbitrary location of the neutral plane and specific properties for the substrate.

I. READ CONFIGURATION FILE

$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(\mathrm{CC2L\_config})$

$a = 0.07$ $\quad b = 0.1429$ $\quad tc = 0.0025$ $\quad vc := \dfrac{cd12}{cd11}$ $\quad vc = 0.485282$

$v := 0.5 \cdot (vc + vsub)$ $\quad v = 0.427641$

II. MODE SOLUTION

In this section, the Rayleigh-Ritz technique is used to find the fundamental eigenvalue and eigenfunction for an 8th order polynomial. (N is preset to 8 below.)

$UU := U(N, b, v)$ $\quad TT := T(N, b, v)$

$gvals := \mathrm{genvals}(UU, TT)$

$gvecs := \mathrm{genvecs}(UU, TT)$

$sgcomb := \mathrm{rsort}\left(\mathrm{stack}\left(gvals^{T}, gvecs\right), 0\right)$

$fundval := \sqrt{sgcomb_{0,0}}$ $\quad fundval = 4.692896$

III. FIND THE COEFFICIENTS FOR THE DEFLECTION FUNCTION

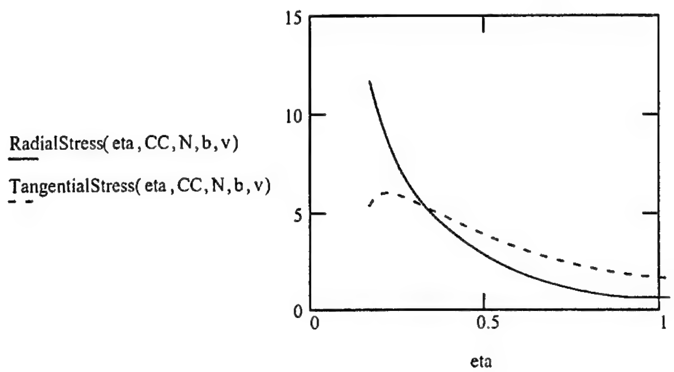
$fundcoef := \mathrm{submatrix}(sgcomb, 1, N, 0, 0)$

$CC := \mathrm{scaledcoef}(fundcoef, N, b)$

The coefficients are normalized so that the maximum deflection is one.

IV. PLOT THE NORMALIZED STRESSES

$eta := b, b + 0.002 .. 1$



V. CALCULATE THE SHAPE INTEGRALS

$Ike := CC^T \cdot TT \cdot CC$	$Ike = 0.206775$
$Ia := 1 - b^2$	$Ia = 0.97958$
$Iv := CC^T \cdot VV(N, b)$	$Iv = 0.569635$
$Iq := CC^T \cdot QQ(N, b)$	$Iq = 1.221375$
$Ipsum := CC^T \cdot Usum(N, b, v) \cdot CC$	$Ipsum = 5.407692$
$Ip1 := CC^T \cdot Up1(N, b, v) \cdot CC$	$Ip1 = 1.491756$
$Ip0 := Ipsum - Ip1$	$Ip0 = 3.915936$

The maximum strain is at the inner radius. The second derivative of the normalized deflection is calculated below (maxcurve) and the peak deflection (i.e., deflection at r = a) corresponding to a strain of one is saved (wnmax).

$maxcurve := CC^T \cdot SM(N, b)$ $\qquad wnmax := \dfrac{a^2}{(tc + z0) \cdot maxcurve_0}$ $\qquad wnmax = 0.136099$

Write shape-integral file:

$WRITEPRN(CC2L\_deflect) := \left(Ip0_0 \;\; Ip1_0 \;\; Iq_0 \;\; Iv_0 \;\; Ike_0 \;\; Ia \;\; wnmax \;\; N \;\; CC \right)$

Because several of the shape integrals are calculated through matrix operations, they remain as matrices even though having dimension 1x1. The subscript index is used to convert to scalar form before writing. The deflection coefficients are written as a matrix (Nx1) as CC.

What follows are predefinitions...

$N \equiv 8$

$$\mathrm{scaledcoef}(vect, N, b) \equiv \left| \begin{array}{l} peakdefl \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad \left| \begin{array}{l} incr \leftarrow 1 - (ii+1)\cdot b^{ii} + ii \cdot b^{ii+1} \\ peakdefl \leftarrow peakdefl + incr \cdot vect_{ii-1} \end{array} \right. \\ vectscaled \leftarrow \dfrac{vect}{peakdefl} \\ vectscaled \end{array} \right.$$

$$\mathrm{Usum}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad \left| \begin{array}{l} u1 \leftarrow (m+1)\cdot(n+1)\cdot \dfrac{1 - b^{m+n}}{m+n} \\ u2 \leftarrow -(n+1)\cdot b^{m} \cdot \dfrac{1 - b^{n}}{n} + -(m+1)\cdot b^{n} \cdot \dfrac{1 - b^{m}}{m} \\ ua_{m-1,n-1} \leftarrow (m+1)\cdot(n+1)\cdot \left(u1 + u2 + -b^{m+n} \cdot \ln(b)\right) \end{array} \right. \\ ua \end{array} \right.$$

$$\mathrm{Up1}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad ubb_{m-1,n-1} \leftarrow n\cdot(n+1)\cdot(m+1)\cdot \left[\dfrac{1 - b^{m+n}}{m+n} - \dfrac{b^{m}\cdot\left(1 - b^{n}\right)}{n} \right] \\ ub \leftarrow ubb + ubb^{T} \\ ub \end{array} \right.$$

$$U(N, b, v) \equiv \left| \begin{array}{l} uu \leftarrow \mathrm{Usum}(N, b, v) - (1 - v)\cdot \mathrm{Up1}(N, b, v) \\ uu \end{array} \right.$$

$$\mathrm{SM}(N, b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad sm_{m-1} \leftarrow m\cdot(m+1)\cdot b^{m-1} \\ sm \end{array} \right.$$

$$T(N,b,v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad \left| \begin{array}{l} t1 \leftarrow \dfrac{1-b^{m+n+4}}{m+n+4} - (n+1)\cdot b^{n}\cdot\dfrac{1-b^{m+4}}{m+4} \\ t2 \leftarrow n\cdot b^{n+1}\cdot\dfrac{1-b^{m+3}}{m+3} - (m+1)\cdot b^{m}\cdot\dfrac{1-b^{n+4}}{n+4} \\ t3 \leftarrow (m+1)\cdot(n+1)\cdot b^{m+n}\cdot\dfrac{1-b^{4}}{4} - (2\cdot m\cdot n + m + n)\cdot b^{m+n+1}\cdot\dfrac{1-b^{3}}{3} \\ t4 \leftarrow m\cdot b^{m+1}\cdot\dfrac{1-b^{n+3}}{n+3} + m\cdot n\cdot b^{m+n+2}\cdot\dfrac{1-b^{2}}{2} \\ tt_{m-1,n-1} \leftarrow t1 + t2 + t3 + t4 \end{array} \right. \\ tt \end{array} \right.$$

$$VV(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow \left[\dfrac{1-b^{m+3}}{m+3} - (m+1)\cdot b^{m}\cdot\dfrac{1-b^{3}}{3}\right] + m\cdot b^{m+1}\cdot\dfrac{1-b^{2}}{2} \\ 2\cdot matv \end{array} \right.$$

$$QQ(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow (m+1)\cdot\left(1-b^{m}\right) \\ matv \end{array} \right.$$

$$\text{slope}(eta, vect, N, b) \equiv \left| \begin{array}{l} slope \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad slope \leftarrow slope + vect_{ii-1}\cdot(ii+1)\cdot\left(eta^{ii} - b^{ii}\right) \\ slope \end{array} \right.$$

$$\text{curve}(eta, vect, N, b) \equiv \left| \begin{array}{l} curve \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad curve \leftarrow curve + vect_{ii-1}\cdot(ii+1)\cdot ii\cdot eta^{ii-1} \\ curve \end{array} \right.$$

$$\text{RadialStress}(eta, vect, N, b, v) \equiv \text{curve}(eta, vect, N, b) + \frac{v\cdot \text{slope}(eta, vect, N, b)}{eta}$$

$$\mathrm{TangentialStress}(\mathrm{eta}, \mathrm{vect}, N, b, v) \equiv v \cdot \mathrm{curve}(\mathrm{eta}, \mathrm{vect}, N, b) + \frac{\mathrm{slope}(\mathrm{eta}, \mathrm{vect}, N, b)}{\mathrm{eta}}$$

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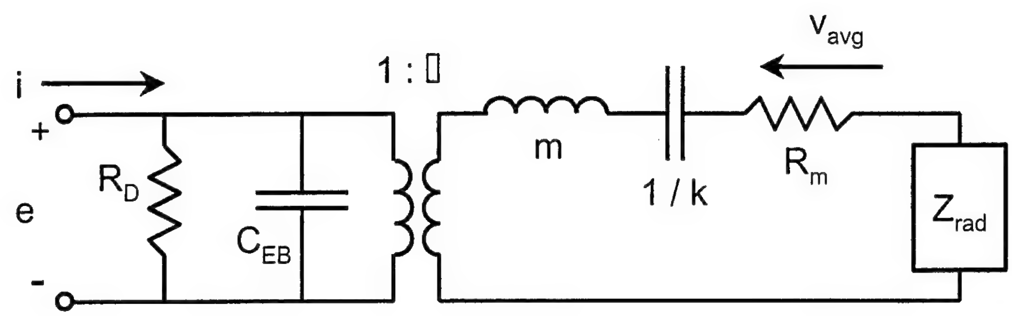
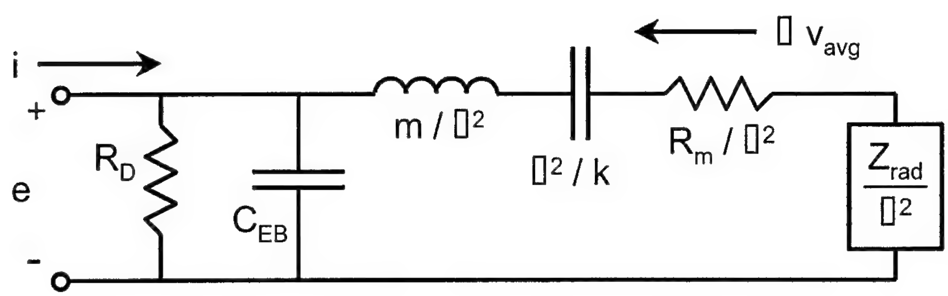
Module 3: Equivalent-Circuit Parameters - Bilaminar Flex Disks

This worksheet reads the configuration and shape-integral files for a particular case and computes the equivalent-circuit parameters.



Alternate form:



In these equivalent circuits, the mechanical quantities, force (pA or acoustic pressure times effective area) and velocity, v, are related to the electrical quantities, voltage, e, and current, i. C_{EB} is the blocked electrical capacitance and R_D is the dielectric loss ($\tan\delta/\omega{*}C_{EB}$). The mechanical mass, stiffness, and mechanical damping are m, k, and R_m respectively. The mechanical resistance is computed based on the mechanical Q of the ceramic and the fraction of strain energy that is stored in the ceramic layer of the composite. The radiation impedance, Zrad, is represented as a parallel combination of resistance and mass (see below). Finally, the electromechanical transduction factor (the "turns ratio") is ϕ.

I. READ FILES

Read configuration file:

$$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN(CC2L\_config)}$$

$a = 0.07$ $\qquad$ $b = 0.143$ $\qquad$ $tc = 2.5 \cdot 10^{-3}$

Read shape-integral file:

$$(\mathrm{Ip0}\ \ \mathrm{Ip1}\ \ \mathrm{Iq}\ \ \mathrm{Iv}\ \ \mathrm{Ike}\ \ \mathrm{Ia}\ \ \mathrm{wnmax}\ \ \mathrm{N}\ \ \mathrm{CC}) := \mathrm{READPRN}(\mathrm{CC2L\_deflect})$$

$$\mathrm{Aeff} := \pi \cdot a^2 \cdot \mathrm{Ia} \qquad \mathrm{Aeff} = 0.015$$

$$\mathrm{ac1} := 1 + 2 \cdot \frac{z0}{tc} \qquad \mathrm{ac2} := 1 + 3 \cdot \frac{z0}{tc} + 3 \cdot \left(\frac{z0}{tc}\right)^2 \qquad \mathrm{as2} := 1 - 3 \cdot \frac{z0}{tsub} + 3 \cdot \left(\frac{z0}{tsub}\right)^2$$

II. CALCULATE EQUIVALENT-CIRCUIT PARAMETERS

Equivalent mass:

$$\mathrm{mass} := 2 \cdot \pi \cdot a^2 \cdot \mathrm{Ike} \cdot \frac{\mathrm{Ia}^2}{\mathrm{Iv}^2} \cdot (\rho c \cdot tc + \rho sub \cdot tsub) \qquad \mathrm{mass} = 0.998$$

Equivalent stiffness:

$$f11 := cd11 - 0.75 \cdot \varepsilon s33 \cdot h31^2 \cdot \frac{\mathrm{ac1}^2}{\mathrm{ac2}} \qquad f12 := cd12 - 0.75 \cdot \varepsilon s33 \cdot h31^2 \cdot \frac{\mathrm{ac1}^2}{\mathrm{ac2}}$$

$$cs11 := \frac{\mathrm{Esub}}{1 - \mathrm{vsub}^2} \qquad cs12 := \mathrm{vsub} \cdot cs11$$

$$\mathrm{Iff} := f11 \cdot \mathrm{Ip0} + f12 \cdot \mathrm{Ip1} \qquad \mathrm{Iss} = cs11 \cdot \mathrm{Ip0} + cs12 \cdot \mathrm{Ip1}$$

$$\mathrm{stiffness} := \frac{2 \cdot \pi}{3 \cdot a^2} \cdot \frac{\mathrm{Ia}^2}{\mathrm{Iv}^2} \cdot \left(tc^3 \cdot \mathrm{ac2} \cdot \mathrm{Iff} + tsub^3 \cdot \mathrm{as2} \cdot \mathrm{Iss}\right) \qquad \mathrm{stiffness} = 4.157 \cdot 10^7$$

Transduction factor:

$$\phi := -\left(\pi \cdot h31 \cdot \varepsilon s33 \cdot \mathrm{Iq} \cdot \frac{\mathrm{Ia}}{\mathrm{Iv}} \cdot tc \cdot \mathrm{ac1}\right) \qquad \phi = 0.392$$

Blocked electrical capacitance:

$$\mathrm{CEB} := \frac{\varepsilon s33 \cdot \pi \cdot a^2}{tc} \cdot \mathrm{Ia} \qquad \mathrm{CEB} = 4.766 \cdot 10^{-8}$$

Coupling factor:

$$kf1 := stiffness \cdot \frac{Iv^2}{Ia^2} \qquad kf2 := \pi \cdot \varepsilon s33 \cdot h31^2 \cdot tc^3 \cdot \frac{Iq^2}{Ia} \cdot \frac{ac1^2}{a^2}$$

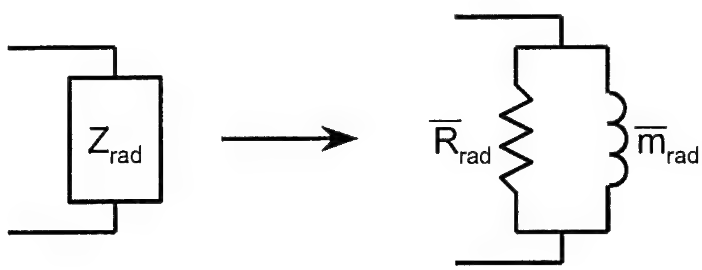
$$\kappa2 := \frac{kf2}{kf1 + kf2} \qquad \kappa2 = 0.072$$

Calculate equivalent Q for composite structure:

$$Uratio := \left(\frac{tsub}{tc}\right)^3 \cdot \frac{as2}{ac2} \cdot \frac{Iss}{Iff} \qquad Qeff := Qmech \cdot (1 + Uratio)$$

$$Rmech := \sqrt{\frac{stiffness}{mass}} \cdot \frac{mass}{Qeff}$$

Calculate circuit elements for parallel form of radiation impedance. This form is preferred over the series form since, in the parallel form, the resistance element is frequency independent. Furthermore, the parallel form degrades more gracefully for ka approaching (or greater than) one. Also, the assumption is made that these transducer elements would always be used in pairs placed back-to-back and driven in phase. In this drive mode, the radiation impedance is equivalent to that of a single element in an infinite rigid baffle.



$$\rho\_water := 1000 \qquad c\_water := 1500$$

$$m\_rad := \frac{8}{3 \cdot \pi} \cdot \rho\_water \cdot a \cdot \left(\pi \cdot a^2\right) \cdot Ia \qquad R\_rad := 1.44 \cdot \rho\_water \cdot c\_water \cdot \left(\pi \cdot a^2\right) \cdot Ia$$

Write equivalent-circuit parameters to a file:

$$WRITEPRN(CC2L\_equivckt) := (mass \;\; stiffness \;\; \phi \;\; CEB \;\; \kappa2 \;\; \tan\delta \;\; Rmech \;\; R\_rad \;\; m\_rad \;\; Aeff)$$

III. MISCELLANEOUS CALCULATIONS

Unloaded resonance frequency:

$$\text{mech\_res} := \frac{1}{2\cdot\pi}\cdot\sqrt{\frac{\text{stiffness}}{\text{mass}}} \qquad \text{mech\_res} = 1.027\cdot 10^{3}$$

Rough estimate of water loading (piston with uniform face velocity):

$$\text{rad\_mass} := \frac{8}{3\cdot\pi}\cdot\rho\text{\_water}\cdot\pi\cdot a^{3}\cdot \text{Ia} \qquad \omega 2\text{water} := \frac{\text{stiffness}}{\text{mass} + \text{rad\_mass}} \qquad \frac{\rho\text{\_water}\cdot a}{\rho c\cdot 2\cdot tc} = 1.842$$

$$\text{mech\_res\_water} := \frac{1}{2\cdot\pi}\cdot\sqrt{\omega 2\text{water}} \qquad \text{mech\_res\_water} = 745.628$$

$$\text{wavenum} := \frac{2\cdot\pi\cdot\text{mech\_res\_water}}{\text{c\_water}} \qquad \text{ka} := \text{wavenum}\cdot a \qquad \text{ka} = 0.219$$

$$\text{radR} := \rho\text{\_water}\cdot\text{c\_water}\cdot\pi\cdot a^{2}\cdot\text{Ia}\cdot\frac{\text{ka}^{2}}{2} \qquad \text{radR} = 540.586$$

Estimated maximum source level (if strain limited):

[Note: Maximum strain is not an appropriate failure criterion for a ceramic material. It is used in this draft version as a crude estimate of limiting performance. Eventually, the more appropriate maximum tensile stress criterion will be implemented. The often-used von Mises stress criterion is also inappropriate for failure in ceramic.]

Enter maximum allowable strain $\quad \text{Smax} := 0.001$

$$\text{Qmax} := \pi\cdot a^{2}\cdot 2\cdot\pi\cdot\text{mech\_res\_water}\cdot\text{wnmax}\cdot\text{Smax}\cdot\text{Iv}$$

$$\text{p1} := \rho\text{\_water}\cdot\text{mech\_res\_water}\cdot\text{Qmax} \qquad \text{SL} := 20\cdot\log(\text{p1}) + 120$$

Maximum pressure at 1 meter (Pa): $\quad \text{p1} = 4.169\cdot 10^{3}$

Maximum SL in dB with respect to 1 micropascal at one meter: $\quad \text{SL} = 192.4$

Estimated peak TVR:

What follows is an optimistic estimate of the peak transmitting voltage response; mechanical loss in the ceramic is ignored.

$$\text{tvr\_peak} := \rho\text{\_water}\cdot\text{mech\_res\_water}\cdot\pi\cdot a^{2}\cdot\text{Ia}\cdot\frac{\phi}{\text{radR}}$$

TVR in pascals at one meter per volt input: $\quad \text{tvr\_peak} = 8.159$

TVR in dB with respect to 1 micropascal at one meter per volt: $\quad 20\cdot\log(|\text{tvr\_peak}|) + 120 = 138.2$

Low-frequency receive response:

The receiving response is not normally important but can be useful in reciprocity calculations.

$$\mathrm{ffvs} := \frac{\pi \cdot a^2 \cdot \mathrm{Ia}}{\phi} \cdot \left(\frac{1}{1 + \frac{\mathrm{CEB} \cdot \mathrm{stiffness}}{\phi^2}} \right)$$

FFVS in open-circuit volts per pascal: $\mathrm{ffvs} = 2.771 \cdot 10^{-3}$

FFVS in dB with respect to one volt per micropascal: $20 \cdot \log(|\mathrm{ffvs}|) - 120 = -171.1$

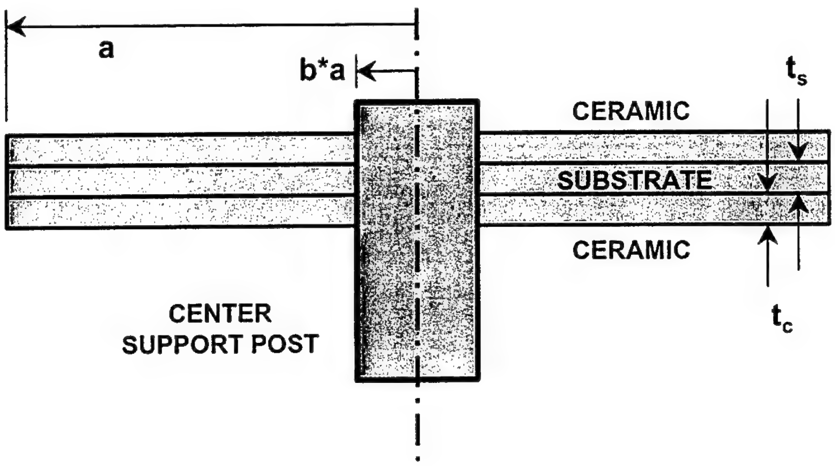
The quantity in square brackets in the equation for ffvs should equal κ^2:

$$\frac{1}{1 + \frac{\mathrm{CEB} \cdot \mathrm{stiffness}}{\phi^2}} = 0.072 \qquad \kappa 2 = 0.072$$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 1.CC3L_config: Configuration Definition - Center-Supported Trilaminar Disk

This worksheet is specific to the annular (center-supported) disk. First, the physical dimensions are specified and the piezoelectric properties file is read. Then, the two-dimensional properties are derived. Finally, the results are written to a configuration file.



I. SELECT MATERIALS [Highlighted values are user inputs]

Choose from PZT4, PZT5H, or PZT8 for file name in pzt READPRN.

$pzt := READPRN(PZT4)$

$se11 := pzt_{0,0} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $se12 := pzt_{0,1} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $\varepsilon t33 := pzt_{0,2} \cdot \varepsilon 0$

$d31 := pzt_{0,3} \cdot 10^{-12} \cdot \frac{m}{volt}$ $\rho := pzt_{0,4} \cdot \frac{kg}{m^3}$ $\tan\delta := pzt_{0,5}$ $Qmech := pzt_{0,6}$

Choose from steel, aluminum, or brass for file name in substrate READPRN.

$substrate := READPRN(brass)$

$\rho sub := substrate_{0,0} \cdot \frac{kg}{m^3}$ $Esub := substrate_{0,1} \cdot 10^9 \cdot Pa$ $vsub := substrate_{0,2}$

$\rho sub = 8.5 \cdot 10^3 \cdot kg \cdot m^{-3}$ $Esub = 1.04 \cdot 10^{11} \cdot kg \cdot m^{-1} \cdot sec^{-2}$ $vsub = 0.37$

II. SELECT DIMENSIONS

Enter dimensions of ceramic disks (a = outside radius; bi = inner radius; tc = thickness) and thickness (tsub) of substrate disk:

[Enter units for in, mm, cm, or m; if no units are entered, meters will be assumed]

$a := 4.0\cdot in \qquad bi := 0.25\cdot in \qquad tc := 0.10\cdot in \qquad tsub := 0.10\cdot in \qquad b := \frac{bi}{a}$

$a = 0.102\cdot m \qquad b = 0.063 \qquad tc = 2.54\cdot 10^{-3}\cdot m \qquad ts2 := \frac{tsub}{2}$

NOTE: For the trilaminar configuration it is assumed that the two ceramic disks are identical.

III. DERIVE PROPERTIES FOR TWO-DIMENSIONAL ANALYSIS

$$sd11 := se11 - \frac{d31^2}{\varepsilon t33} \qquad sd12 := se12 - \frac{d31^2}{\varepsilon t33} \qquad -\frac{sd12}{sd11} = 0.485$$

$$cd11 := \frac{sd11}{sd11^2 - sd12^2} \qquad cd12 := \frac{-sd12}{sd11^2 - sd12^2} \qquad \frac{cd12}{cd11} = 0.485$$

$$\varepsilon s33 := \varepsilon t33 - \frac{2\cdot d31^2}{se11 + se12} \qquad h31 := \frac{d31}{\varepsilon t33\cdot(sd11 + sd12)}$$

IV. WRITE CONFIGURATION FILE

PRN files cannot handle dimensions so all quantities are written (in SI) without units.

$$out_{0,0} := \frac{\rho}{\rho 0} \qquad out_{0,1} := \frac{a}{m0} \qquad out_{0,2} := b \qquad out_{0,3} := \frac{tc}{m0}$$

$$out_{0,4} := \frac{cd11}{c0} \qquad out_{0,5} := \frac{cd12}{c0} \qquad out_{0,6} := \frac{h31}{h0} \qquad out_{0,7} := \frac{\varepsilon s33}{e0}$$

$$out_{0,8} := \tan\delta \qquad out_{0,9} := Qmech \qquad out_{0,10} := \frac{\rho sub}{\rho 0} \qquad out_{0,11} := \frac{Esub}{c0}$$

$$out_{0,12} := vsub \qquad out_{0,13} := \frac{ts2}{m0} \qquad out_{0,14} := \frac{ts2}{m0}$$

WRITEPRN(CC3L_config) := out

The configuration file contains the following quantities in this order:

density of ceramic, ρ
outer radius of disk, a
ratio of inner radius to outer radius, b
thickness of ceramic, t
cd11 for ceramic
cd12 for ceramic
h31 for ceramic
εs33 for ceramic
loss tangent for ceramic, tanδ
mechanical Q of ceramic
density of substrate, ρsub
modulus of substrate, Esub
poisson's ratio of substrate, vsub
half thickness of substrate, ts2
location of neutral plane (same as half substrate thickness)

Unit normalization quantities (predefined):

$\rho 0 \equiv 1 \cdot \frac{kg}{m^3}$ $\quad m0 \equiv 1 \cdot m$ $\quad c0 \equiv 1 \cdot Pa$ $\quad h0 \equiv 1 \cdot \frac{volt}{m}$ $\quad e0 \equiv 1 \cdot \frac{farad}{m}$

$\varepsilon 0 \equiv 8.854 \cdot 10^{-12} \cdot \frac{farad}{m}$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 2.CC3L_deflect: Center-Supported Trilaminar Disk -- Clamped Inside

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I. READ CONFIGURATION FILE

$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(CC3L\_config)$

$a = 0.1016$ $\quad b = 0.0625$ $\quad tc = 0.00254$ $\quad vc := \dfrac{cd12}{cd11}$ $\quad vc = 0.485282$

$v := 0.5\cdot(vc + vsub)$ $\quad v = 0.427641$

II. MODE SOLUTION

In this section, the Rayleigh-Ritz technique is used to find the fundamental eigenvalue and eigenfunction for an 8th order polynomial. (N is preset to 8 below.)

$UU := U(N, b, v)$ $\quad TT := T(N, b, v)$

$gvals := \mathrm{genvals}(UU, TT)$

$gvecs := \mathrm{genvecs}(UU, TT)$

$sgcomb := \mathrm{rsort}\left(\mathrm{stack}\left(gvals^{T}, gvecs\right), 0\right)$

$fundval := \sqrt{sgcomb_{0,0}}$ $\quad fundval = 4.078196$

III. FIND THE COEFFICIENTS FOR THE DEFLECTION FUNCTION

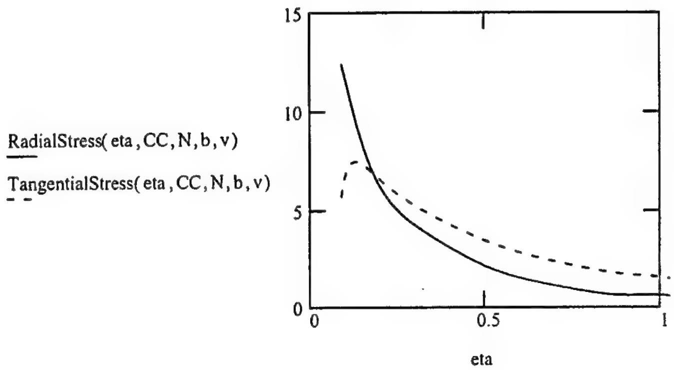
$fundcoef := \mathrm{submatrix}(sgcomb, 1, N, 0, 0)$

$CC := \mathrm{scaledcoef}(fundcoef, N, b)$

The coefficients are normalized so that the maximum deflection is one.

IV. PLOT THE NORMALIZED STRESSES

eta := b, b + 0.002.. 1



V. CALCULATE THE SHAPE INTEGRALS

$Ike := CC^T \cdot TT \cdot CC$	$Ike = 0.225435$
$Ia := 1 - b^2$	$Ia = 0.996094$
$Iv := CC^T \cdot VV(N, b)$	$Iv = 0.611179$
$Iq := CC^T \cdot QQ(N, b)$	$Iq = 1.076708$
$Ipsum := CC^T \cdot Usum(N, b, v) \cdot CC$	$Ipsum = 4.412892$
$Ip1 := CC^T \cdot Up1(N, b, v) \cdot CC$	$Ip1 = 1.1593$
$Ip0 := Ipsum - Ip1$	$Ip0 = 3.253593$

The maximum strain is at the inner radius. The second derivative of the normalized deflection is calculated below (maxcurve) and the peak deflection (i.e., deflection at r = a) corresponding to a strain of one is saved (wnmax).

$maxcurve := CC^T \cdot SM(N, b)$ $\qquad wnmax := \dfrac{a^2}{(tc + z0) \cdot maxcurve_0}$ $\qquad wnmax = 0.230222$

Write shape-integral file:

$WRITEPRN(CC3L\_deflect) := \left(Ip0_0 \;\; Ip1_0 \;\; Iq_0 \;\; Iv_0 \;\; Ike_0 \;\; Ia \;\; wnmax \;\; N \;\; CC\right)$

Because several of the shape integrals are calculated through matrix operations, they remain as matrices even though having dimension 1x1. The subscript index is used to convert to scalar form before writing. The deflection coefficients are written as a matrix (Nx1) as CC.

What follows are predefinitions...

$N \equiv 8$

$$
\text{scaledcoef}(\text{vect},N,b) \equiv \left|
\begin{array}{l}
\text{peakdefl} \leftarrow 0 \\
\text{for } ii \in 1..N \\
\quad \left| \begin{array}{l}
\text{incr} \leftarrow 1 - (ii+1)\cdot b^{ii} + ii\cdot b^{ii+1} \\
\text{peakdefl} \leftarrow \text{peakdefl} + \text{incr}\cdot \text{vect}_{ii-1}
\end{array} \right. \\
\text{vectscaled} \leftarrow \dfrac{\text{vect}}{\text{peakdefl}} \\
\text{vectscaled}
\end{array} \right.
$$

$$
\text{Usum}(N,b,v) \equiv \left|
\begin{array}{l}
\text{for } m \in 1..N \\
\quad \text{for } n \in 1..N \\
\quad\quad \left| \begin{array}{l}
u1 \leftarrow (m+1)\cdot(n+1)\cdot\dfrac{1-b^{m+n}}{m+n} \\
u2 \leftarrow -(n+1)\cdot b^{m}\cdot\dfrac{1-b^{n}}{n} + -(m+1)\cdot b^{n}\cdot\dfrac{1-b^{m}}{m} \\
ua_{m-1,n-1} \leftarrow (m+1)\cdot(n+1)\cdot\left(u1 + u2 + -b^{m+n}\cdot\ln(b)\right)
\end{array} \right. \\
ua
\end{array} \right.
$$

$$
\text{Up1}(N,b,v) \equiv \left|
\begin{array}{l}
\text{for } m \in 1..N \\
\quad \text{for } n \in 1..N \\
\quad\quad ubb_{m-1,n-1} \leftarrow n\cdot(n+1)\cdot(m+1)\cdot\left[\dfrac{1-b^{m+n}}{m+n} - \dfrac{b^{m}\cdot\left(1-b^{n}\right)}{n}\right] \\
ub \leftarrow ubb + ubb^{T} \\
ub
\end{array} \right.
$$

$$
U(N,b,v) \equiv \left|
\begin{array}{l}
uu \leftarrow \text{Usum}(N,b,v) - (1-v)\cdot \text{Up1}(N,b,v) \\
uu
\end{array} \right.
$$

$$
\text{SM}(N,b) \equiv \left|
\begin{array}{l}
\text{for } m \in 1..N \\
\quad sm_{m-1} \leftarrow m\cdot(m+1)\cdot b^{m-1} \\
sm
\end{array} \right.
$$

$$
T(N,b,v) \equiv \left|\begin{array}{l}
\text{for } m \in 1..N \\
\quad \text{for } n \in 1..N \\
\quad\quad \left|\begin{array}{l}
t1 \leftarrow \dfrac{1-b^{m+n+4}}{m+n+4} - (n+1)\cdot b^{n}\cdot\dfrac{1-b^{m+4}}{m+4} \\[2ex]
t2 \leftarrow n\cdot b^{n+1}\cdot\dfrac{1-b^{m+3}}{m+3} - (m+1)\cdot b^{m}\cdot\dfrac{1-b^{n+4}}{n+4} \\[2ex]
t3 \leftarrow (m+1)\cdot(n+1)\cdot b^{m+n}\cdot\dfrac{1-b^{4}}{4} - (2\cdot m\cdot n+m+n)\cdot b^{m+n+1}\cdot\dfrac{1-b^{3}}{3} \\[2ex]
t4 \leftarrow m\cdot b^{m+1}\cdot\dfrac{1-b^{n+3}}{n+3} + m\cdot n\cdot b^{m+n+2}\cdot\dfrac{1-b^{2}}{2} \\[2ex]
tt_{m-1,n-1} \leftarrow t1+t2+t3+t4
\end{array}\right. \\
tt
\end{array}\right.
$$

$$
VV(N,b) \equiv \left|\begin{array}{l}
\text{for } m \in 1..N \\
\quad matv_{m-1} \leftarrow \left[\dfrac{1-b^{m+3}}{m+3} - (m+1)\cdot b^{m}\cdot\dfrac{1-b^{3}}{3}\right] + m\cdot b^{m+1}\cdot\dfrac{1-b^{2}}{2} \\
2\cdot matv
\end{array}\right.
$$

$$
QQ(N,b) \equiv \left|\begin{array}{l}
\text{for } m \in 1..N \\
\quad matv_{m-1} \leftarrow (m+1)\cdot\left(1-b^{m}\right) \\
matv
\end{array}\right.
$$

$$
\text{slope}(eta, vect, N, b) \equiv \left|\begin{array}{l}
slope \leftarrow 0 \\
\text{for } ii \in 1..N \\
\quad slope \leftarrow slope + vect_{ii-1}\cdot(ii+1)\cdot\left(eta^{ii} - b^{ii}\right) \\
slope
\end{array}\right.
$$

$$
\text{curve}(eta, vect, N, b) \equiv \left|\begin{array}{l}
curve \leftarrow 0 \\
\text{for } ii \in 1..N \\
\quad curve \leftarrow curve + vect_{ii-1}\cdot(ii+1)\cdot ii\cdot eta^{ii-1} \\
curve
\end{array}\right.
$$

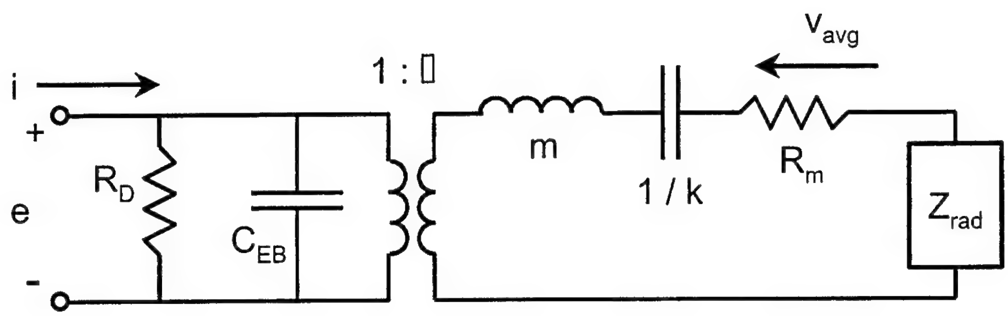
$$
\text{RadialStress}(eta, vect, N, b, v) \equiv \text{curve}(eta, vect, N, b) + \frac{v\cdot \text{slope}(eta, vect, N, b)}{eta}
$$

$$
\text{TangentialStress}(eta, vect, N, b, v) \equiv v\cdot \text{curve}(eta, vect, N, b) + \frac{\text{slope}(eta, vect, N, b)}{eta}
$$

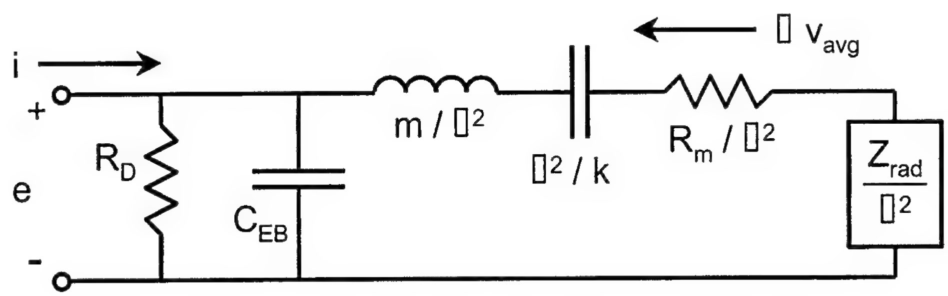
Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 3: Equivalent-Circuit Parameters -- Trilaminar Flex Disks

This worksheet reads the configuration and shape-integral files for a particular case and computes the equivalent-circuit parameters.



Alternate form:



In these equivalent circuits, the mechanical quantities, force (pA or acoustic pressure times effective area) and velocity, v, are related to the electrical quantities, voltage, e, and current, i. C_{EB} is the blocked electrical capacitance and R_D is the dielectric loss ($\tan\delta/\omega^* C_{EB}$). The mechanical mass, stiffness, and mechanical damping are m, k, and R_m respectively. The mechanical resistance is computed based on the mechanical Q of the ceramic and the fraction of strain energy that is stored in the ceramic layer of the composite. The radiation impedance, Zrad, is represented as a parallel combination of resistance and mass (see below). Finally, the electromechanical transduction factor (the "turns ratio") is ϕ.

I. READ FILES

Read configuration file:

$$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(CC3L\_config)$$

$a = 0.102$ $\qquad$ $b = 0.063$ $\qquad$ $tc = 2.54 \cdot 10^{-3}$

Read shape-integral file:

$$(Ip0\ \ Ip1\ \ Iq\ \ Iv\ \ Ike\ \ Ia\ \ wnmax\ \ N\ \ CC) := \mathrm{READPRN}(CC3L\_deflect)$$

$$ac1 := 1 + 2\cdot\frac{z0}{tc} \qquad ac2 := 1 + 3\cdot\frac{z0}{tc} + 3\cdot\left(\frac{z0}{tc}\right)^2 \qquad as2 := 1 - 3\cdot\frac{z0}{tsub} + 3\cdot\left(\frac{z0}{tsub}\right)^2$$

$$Aeff := \pi\cdot a^2\cdot Ia \qquad Aeff = 0.032$$

II. CALCULATE EQUIVALENT-CIRCUIT PARAMETERS

Equivalent mass:

$$mass := 4\cdot\pi\cdot a^2\cdot Ike\cdot\frac{Ia^2}{Iv^2}\cdot(\rho c\cdot tc + \rho sub\cdot tsub) \qquad mass = 2.338$$

Equivalent stiffness:

$$f11 := cd11 - 0.75\cdot\varepsilon s33\cdot h31^2\cdot\frac{ac1^2}{ac2} \qquad f12 := cd12 - 0.75\cdot\varepsilon s33\cdot h31^2\cdot\frac{ac1^2}{ac2}$$

$$cs11 := \frac{Esub}{1 - vsub^2} \qquad cs12 := vsub\cdot cs11$$

$$Iff := f11\cdot Ip0 + f12\cdot Ip1 \qquad Iss := cs11\cdot Ip0 + cs12\cdot Ip1$$

$$stiffness := \frac{4\cdot\pi}{3\cdot a^2}\cdot\frac{Ia^2}{Iv^2}\cdot\left(tc^3\cdot ac2\cdot Iff + tsub^3\cdot as2\cdot Iss\right) \qquad stiffness = 2.056\cdot 10^7$$

Transduction factor for parallel connection of ceramic layers:

$$\phi := -2\cdot\left(\pi\cdot h31\cdot\varepsilon s33\cdot Iq\cdot\frac{Ia}{Iv}\cdot tc\cdot ac1\right) \qquad \phi = 0.828$$

Blocked electrical capacitance for parallel connection of ceramic layers:

$$CEB := \frac{2\cdot\varepsilon s33\cdot\pi\cdot a^2}{tc}\cdot Ia \qquad CEB = 2.01\cdot 10^{-7}$$

Coupling factor (for parallel connection of ceramic layers):

$$kf1 := stiffness\cdot\frac{Iv^2}{Ia^2} \qquad kf2 := 2\cdot\pi\cdot\varepsilon s33\cdot h31^2\cdot tc^3\cdot\frac{Iq^2}{Ia}\cdot\frac{ac1^2}{a^2}$$

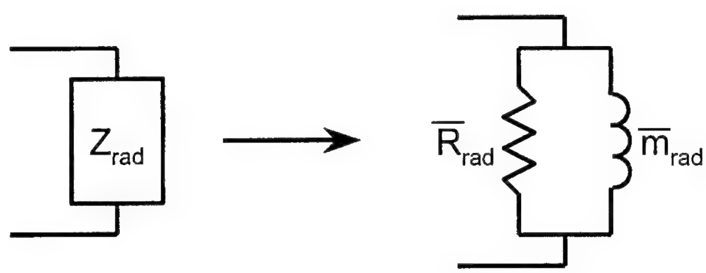
$$\kappa 2 := \frac{kf2}{kf1 + kf2} \qquad \kappa 2 = 0.142$$

Calculate equivalent Q for composite structure:

$$Uratio := \left(\frac{tsub}{tc}\right)^3 \cdot \frac{as2}{ac2} \cdot \frac{Iss}{Iff} \qquad Qeff := Qmech \cdot (1 + Uratio)$$

$$Rmech := \sqrt{\frac{stiffness}{mass}} \cdot \frac{mass}{Qeff}$$

Calculate circuit elements for parallel form of radiation impedance. This form is preferred over the series form since, in the parallel form, the resistance element is frequency independent. Furthermore, the parallel form degrades more gracefully for ka approaching (or greater than) one. Also, the assumption is made that these transducer elements would always be used in pairs placed back-to-back and driven in phase. In this drive mode, the radiation impedance is equivalent to that of a single element in an infinite rigid baffle.



$$\rho\_water := 1000 \qquad c\_water := 1500$$

$$m\_rad := \frac{8}{3 \cdot \pi} \cdot \rho\_water \cdot a \cdot \left(\pi \cdot a^2\right) \cdot Ia \qquad R\_rad := 1.44 \cdot \rho\_water \cdot c\_water \cdot \left(\pi \cdot a^2\right) \cdot Ia$$

Write equivalent-circuit parameters to a file:

$$WRITEPRN(CC3L\_equivckt) := (mass \;\; stiffness \;\; \phi \;\; CEB \;\; \kappa 2 \;\; \tan\delta \;\; Rmech \;\; R\_rad \;\; m\_rad \;\; Aeff)$$

III. MISCELLANEOUS CALCULATIONS

Unloaded resonance frequency:

$$mech\_res := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{stiffness}{mass}} \qquad mech\_res = 471.953$$

Rough estimate of water loading (piston with uniform face velocity):

$$\text{rad\_mass} := \frac{8}{3\cdot\pi}\cdot\rho\text{\_water}\cdot\pi\cdot a^3\cdot \text{Ia} \qquad \omega2\text{water} := \frac{\text{stiffness}}{\text{mass} + \text{rad\_mass}} \qquad \frac{\rho\text{\_water}\cdot a}{\rho c\cdot 2\cdot tc} = 2.632$$

$$\text{mech\_res\_water} := \frac{1}{2\cdot\pi}\cdot\sqrt{\omega2\text{water}} \qquad \text{mech\_res\_water} = 318.803$$

$$\text{wavenum} := \frac{2\cdot\pi\cdot\text{mech\_res\_water}}{\text{c\_water}} \qquad \text{ka} := \text{wavenum}\cdot a \qquad \text{ka} = 0.136$$

$$\text{radR} := \rho\text{\_water}\cdot\text{c\_water}\cdot\pi\cdot a^2\cdot\text{Ia}\cdot\frac{\text{ka}^2}{2} \qquad \text{radR} = 445.971$$

Estimated maximum source level (if strain limited):

[Note: Maximum strain is not an appropriate failure criterion for a ceramic material. It is used in this draft version as a crude estimate of limiting performance. Eventually, the more appropriate maximum tensile stress criterion will be implemented. The often-used von Mises stress criterion is also inappropriate for failure in ceramic.]

Enter maximum allowable strain $\text{Smax} := 0.001$

$$\text{Qmax} := \pi\cdot a^2\cdot 2\cdot\pi\cdot\text{mech\_res\_water}\cdot\text{wnmax}\cdot\text{Smax}\cdot\text{Iv}$$

$$\text{p1} := \rho\text{\_water}\cdot\text{mech\_res\_water}\cdot\text{Qmax} \qquad \text{SL} := 20\cdot\log(\text{p1}) + 120$$

Maximum pressure at 1 meter (Pa): $\text{p1} = 2.914\cdot 10^3$

Maximum SL in dB with respect to 1 micropascal at one meter: $\text{SL} = 189.3$

Estimated peak TVR:

What follows is an optimistic estimate of the peak transmitting voltage response; mechanical loss in the ceramic is ignored.

$$\text{tvr\_peak} := \rho\text{\_water}\cdot\text{mech\_res\_water}\cdot\pi\cdot a^2\cdot\text{Ia}\cdot\frac{\phi}{\text{radR}}$$

TVR in pascals at one meter per volt input: $\text{tvr\_peak} = 19.122$

TVR in dB with respect to 1 micropascal at one meter per volt: $20\cdot\log(|\text{tvr\_peak}|) + 120 = 145.6$

Low-frequency receive response:

The receiving response is not normally important but can be useful in reciprocity calculations.

$$\mathrm{ffvs} := \frac{\pi \cdot a^2 \cdot \mathrm{Ia}}{\phi} \cdot \left(\frac{1}{1 + \frac{\mathrm{CEB} \cdot \mathrm{stiffness}}{\phi^2}} \right)$$

FFVS in open-circuit volts per pascal: $\mathrm{ffvs} = 5.552 \cdot 10^{-3}$

FFVS in dB with respect to one volt per micropascal: $20 \cdot \log(|\mathrm{ffvs}|) - 120 = -165.1$

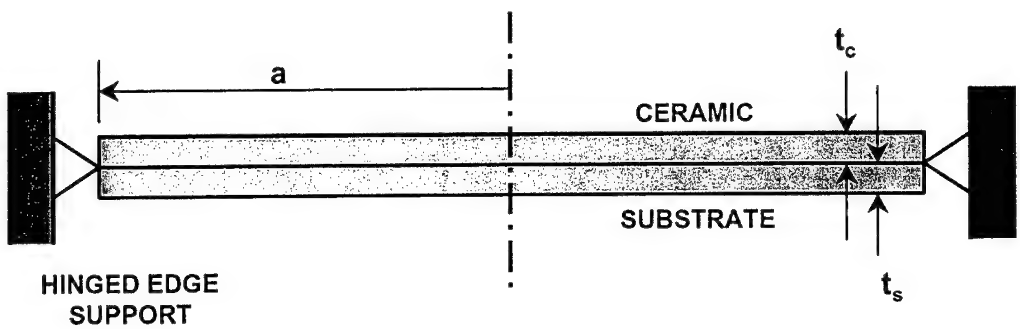
The quantity in square brackets in the equation for ffvs should equal $\square^2$:

$$\frac{1}{1 + \frac{\mathrm{CEB} \cdot \mathrm{stiffness}}{\phi^2}} = 0.142 \qquad \kappa 2 = 0.142$$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 1.ES2L_config: Configuration Definition - Edge-Supported Disk

This worksheet is specific to the bilaminar edge-supported disk. First, the physical dimensions are specified and the piezoelectric properties file is read. Then, the two-dimensional properties are derived. Finally, the results are written to a configuration file.



I. SELECT MATERIALS [Highlighted values are user inputs]

Choose from PZT4, PZT5H, or PZT8 for file name in pzt READPRN.

$pzt := READPRN(PZT4)$

$se11 := pzt_{0,0} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $se12 := pzt_{0,1} \cdot 10^{-12} \cdot \frac{1}{Pa}$ $\varepsilon t33 = pzt_{0,2} \cdot \varepsilon 0$

$d31 := pzt_{0,3} \cdot 10^{-12} \cdot \frac{m}{volt}$ $\rho := pzt_{0,4} \cdot \frac{kg}{m^3}$ $\tan\delta := pzt_{0,5}$ $Qmech := pzt_{0,6}$

Choose from steel, aluminum, or brass for file name in substrate READPRN.

$substrate := READPRN(brass)$

$\rho sub := substrate_{0,0} \cdot \frac{kg}{m^3}$ $Esub := substrate_{0,1} \cdot 10^9 \cdot Pa$ $vsub := substrate_{0,2}$

$\rho sub = 8.5 \cdot 10^3 \ \cdot kg \cdot m^{-3}$ $Esub = 1.04 \cdot 10^{11} \ \cdot kg \cdot m^{-1} \cdot sec^{-2}$ $vsub = 0.37$

II. SELECT DIMENSIONS

Enter dimensions of ceramic disk (a = outside radius; tc = thickness)
and thickness (tsub) of substrate disk:

[Enter units for in, mm, cm, or m; if no units are entered, meters will be assumed]

$a := 10\cdot cm$ $\qquad tc := 0.1\cdot mm$ $\qquad tsub := 1\cdot cm$ $\qquad b := 0$

$a = 0.1\cdot m$ $\qquad b = 0$ $\qquad tc = 1\cdot 10^{-4}\ \cdot m$

III. DERIVE PROPERTIES FOR TWO-DIMENSIONAL ANALYSIS

$$sd11 := se11 - \frac{d31^2}{\varepsilon t33} \qquad sd12 := se12 - \frac{d31^2}{\varepsilon t33} \qquad -\frac{sd12}{sd11} = 0.485$$

$$cd11 := \frac{sd11}{sd11^2 - sd12^2} \qquad cd12 := \frac{-sd12}{sd11^2 - sd12^2} \qquad \frac{cd12}{cd11} = 0.485$$

$$\varepsilon s33 := \varepsilon t33 - \frac{2\cdot d31^2}{se11 + se12} \qquad h31 := \frac{d31}{\varepsilon t33\cdot(sd11 + sd12)}$$

IV. CALCULATE LOCATION OF NEUTRAL PLANE

Origin of z-coordinate at neutral plane; z0 gives distance from neutral plane to ceramic substrate interface:

$$fs := \frac{Esub}{1 - vsub^2} \qquad z0 := \frac{tsub^2\cdot fs - tc^2\cdot cd11}{2\cdot(tsub\cdot fs + tc\cdot cd11)} \qquad z0 = 4.951\cdot 10^{-3}\ \cdot m$$

WARNING: If z0 is negative, then the neutral plane is in ceramic and some part
of the ceramic will be in tension under hydrostatic load unless prestressed.

V. WRITE CONFIGURATION FILE

PRN files cannot handle dimensions so all quantities are written (in SI) without units.

$$out_{0,0} := \frac{\rho}{\rho 0} \qquad out_{0,1} := \frac{a}{m0} \qquad out_{0,2} := b \qquad out_{0,3} := \frac{tc}{m0}$$

$$out_{0,4} := \frac{cd11}{c0} \qquad out_{0,5} := \frac{cd12}{c0} \qquad out_{0,6} := \frac{h31}{h0} \qquad out_{0,7} := \frac{\varepsilon s33}{e0}$$

$$out_{0,8} := \tan\delta \qquad out_{0,9} := Qmech \qquad out_{0,10} := \frac{\rho sub}{\rho 0} \qquad out_{0,11} := \frac{Esub}{c0}$$

$out_{0,12} := vsub$ $out_{0,13} := \frac{tsub}{m0}$ $out_{0,14} := \frac{z0}{m0}$

$WRITEPRN(ES2L\_config) := out$

The configuration file contains the following quantities in this order:

density of ceramic, ρ
outer radius of disk, a
ratio of inner radius to outer radius, b (always zero here)
thickness of ceramic, tc
cd11 for ceramic
cd12 for ceramic
h31 for ceramic
εs33 for ceramic
loss tangent for ceramic, tanδ
mechanical Q of ceramic
density of substrate, ρsub
modulus of substrate, Esub
poisson's ratio of substrate, vsub
thickness of substrate, tsub
z location of neutral plane, z0, with respect to bottom of ceramic

Unit normalization quantities (predefined):

$\rho 0 \equiv 1 \cdot \frac{kg}{m^3}$ $m0 \equiv 1 \cdot m$ $c0 \equiv 1 \cdot Pa$ $h0 \equiv 1 \cdot \frac{volt}{m}$ $e0 \equiv 1 \cdot \frac{farad}{m}$

$\varepsilon 0 \equiv 8.854 \cdot 10^{-12} \cdot \frac{farad}{m}$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 2.ES2L_deflect: Edge-Supported Disk

This worksheet reads the configuration file, calculates the first axisymmetric mode deflection function, and then calculates all of the shape integrals. The shape integrals are written to a file for subsequent use. In this version, the polynomial Rayleigh-Ritz method is used to compute the mode deflection function. An eighth-order polynomial is used. This module is expecting the configuration file from version 2 -- that is, the configuration with arbitrary location of the neutral plane and specific properties for the substrate.

I. READ CONFIGURATION FILE

$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(ES2L\_config)$

$a = 0.1$ $\quad b = 0$ $\quad tc = 0.0001$ $\quad vc := \dfrac{cd12}{cd11}$ $\quad vc = 0.485282$

$v := 0.5\cdot(vc + vsub)$ $\quad v = 0.427641$

II. MODE SOLUTION

In this section, the Rayleigh-Ritz technique is used to find the fundamental eigenvalue and eigenfunction for an 8th order polynomial. (N is preset to 8 below.)

$UU := U(N, b, v)$ $\quad TT := T(N, b, v)$

$gvals := genvals(UU, TT)$

$gvecs := genvecs(UU, TT)$

$sgcomb := rsort\left(stack\left(gvals^{T}, gvecs\right), 0\right)$

$fundval := \sqrt{sgcomb_{0,0}}$ $\quad fundval = 5.116155$

III. FIND THE COEFFICIENTS FOR THE DEFLECTION FUNCTION

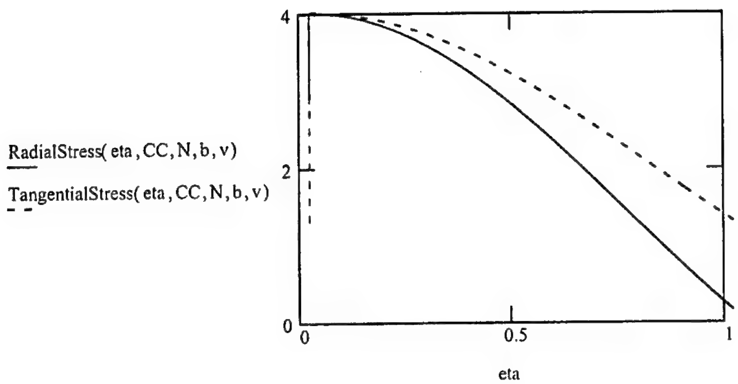
$fundcoef := submatrix(sgcomb, 1, N, 0, 0)$

$CC := scaledcoef(fundcoef, N, b)$

The coefficients are normalized so that the maximum deflection is one.

IV. PLOT THE NORMALIZED STRESSES

$\text{eta} := b, b + 0.002 .. 1$



V. CALCULATE THE SHAPE INTEGRALS

$\text{Ike} := CC^T \cdot TT \cdot CC$	$\text{Ike} = 0.140926$
$\text{Ia} := 1 - b^2$	$\text{Ia} = 1$
$\text{Iv} := CC^T \cdot VV(N, b)$	$\text{Iv} = 0.445882$
$\text{Iq} := CC^T \cdot QQ(N, b)$	$\text{Iq} = 1.401826$
$\text{Ipsum} := CC^T \cdot \text{Usum}(N, b, v) \cdot CC$	$\text{Ipsum} = 4.813486$
$\text{Ip1} := CC^T \cdot \text{Up1}(N, b, v) \cdot CC$	$\text{Ip1} = 1.965116$
$\text{Ip0} := \text{Ipsum} - \text{Ip1}$	$\text{Ip0} = 2.84837$

The maximum strain is at the inner radius. The second derivative of the normalized deflection is calculated below (maxcurve) and the peak deflection (i.e., deflection at r = a) corresponding to a strain of one is saved (wnmax).

$\text{maxcurve} := CC^T \cdot SM(N, b)$ $\qquad \text{wnmax} := \dfrac{a^2}{(tc + z0) \cdot \text{maxcurve}_0}$ $\qquad \text{wnmax} = 0.732203$

Write shape-integral file:

$\text{WRITEPRN(ES2L\_deflect)} := \left(\text{Ip0}_0 \ \ \text{Ip1}_0 \ \ \text{Iq}_0 \ \ \text{Iv}_0 \ \ \text{Ike}_0 \ \ \text{Ia} \ \ \text{wnmax} \ \ N \ \ CC\right)$

Because several of the shape integrals are calculated through matrix operations, they remain as matrices even though having dimension 1x1. The subscript index is used to convert to scalar form before writing. The coefficients are written as a matrix (Nx1) as CC.

What follows are predefinitions...

$N \equiv 8$

$$\mathrm{scaledcoef}(vect, N, b) \equiv \left| \begin{array}{l} peakdefl \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad peakdefl \leftarrow peakdefl + vect_{ii-1} \\ vectscaled \leftarrow \dfrac{vect}{peakdefl} \\ vectscaled \end{array} \right.$$

$$\mathrm{Usum}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \qquad ua_{m-1,n-1} \leftarrow \dfrac{(m+1)^2 \cdot (n+1)^2}{m-n} \\ ua \end{array} \right.$$

$$\mathrm{Up1}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \qquad ubb_{m-1,n-1} \leftarrow \dfrac{n \cdot (n+1) \cdot (m+1)}{m+n} \\ ub \leftarrow ubb + ubb^T \\ ub \end{array} \right.$$

$$U(N, b, v) \equiv \left| \begin{array}{l} uu \leftarrow \mathrm{Usum}(N, b, v) - (1 - v) \cdot \mathrm{Up1}(N, b, v) \\ uu \end{array} \right.$$

$$\mathrm{SM}(N, b) \equiv \left| \begin{array}{l} \text{for } m \in 2..N \\ \quad sm_{m-1} \leftarrow 0 \\ sm_0 \leftarrow 2 \\ sm \end{array} \right.$$

$$T(N,b,v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad tt_{m-1,n-1} \leftarrow \dfrac{1}{m+n+4} - \dfrac{1}{m+3} - \dfrac{1}{n+3} + \dfrac{1}{2} \\ tt \end{array} \right.$$

$$VV(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow 1 - \dfrac{2}{m+3} \\ matv \end{array} \right.$$

$$QQ(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow (m+1) \\ matv \end{array} \right.$$

$$\text{slope}(\text{eta}, \text{vect}, N, b) \equiv \left| \begin{array}{l} \text{slope} \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad \text{slope} \leftarrow \text{slope} + \text{vect}_{ii-1} \cdot (ii+1) \cdot \text{eta}^{ii} \\ \text{slope} \end{array} \right.$$

$$\text{curve}(\text{eta}, \text{vect}, N, b) \equiv \left| \begin{array}{l} \text{curve} \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad \text{curve} \leftarrow \text{curve} + \text{vect}_{ii-1} \cdot (ii+1) \cdot ii \cdot \text{eta}^{ii-1} \\ \text{curve} \end{array} \right.$$

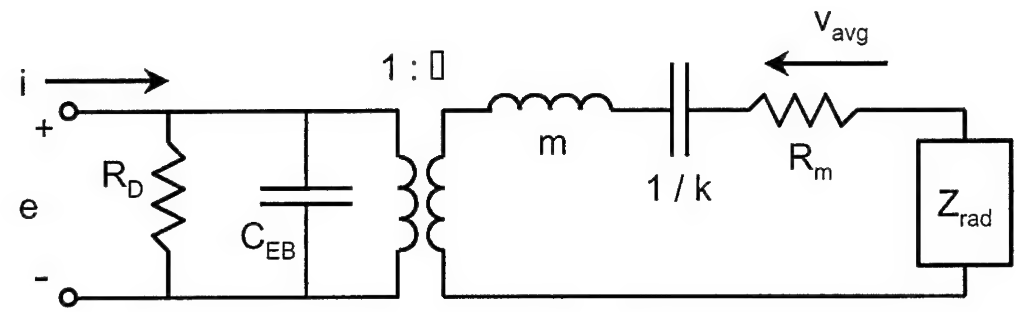
$$\text{RadialStress}(\text{eta}, \text{vect}, N, b, v) \equiv \text{curve}(\text{eta}, \text{vect}, N, b) + \frac{v \cdot \text{slope}(\text{eta}, \text{vect}, N, b)}{\text{eta}}$$

$$\text{TangentialStress}(\text{eta}, \text{vect}, N, b, v) \equiv v \cdot \text{curve}(\text{eta}, \text{vect}, N, b) + \frac{\text{slope}(\text{eta}, \text{vect}, N, b)}{\text{eta}}$$

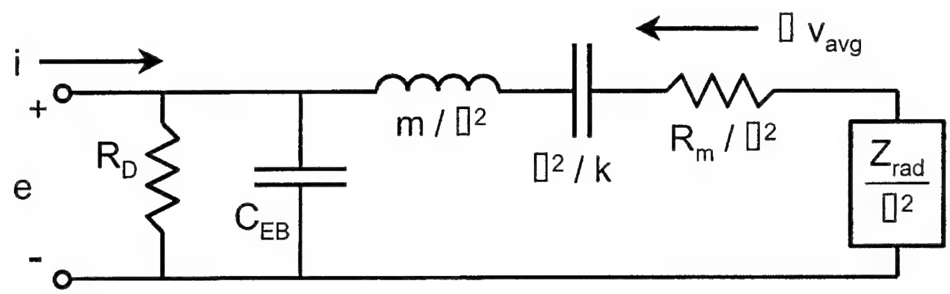
Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 3: Equivalent-Circuit - Bilaminar, Edge-Supported Flex Disks

This worksheet reads the configuration and shape-integral files for a particular case and computes the equivalent-circuit parameters.



Alternate form:



In these equivalent circuits, the mechanical quantities, force (pA or acoustic pressure times effective area) and velocity, v, are related to the electrical quantities, voltage, e, and current, i. C_{EB} is the blocked electrical capacitance and R_D is the dielectric loss ($\tan\delta/\omega^* C_{EB}$). The mechanical mass, stiffness, and mechanical damping are m, k, and R_m respectively. The mechanical resistance is computed based on the mechanical Q of the ceramic and the fraction of strain energy that is stored in the ceramic layer of the composite. The radiation impedance, Zrad, is represented as a parallel combination of resistance and mass (see below). Finally, the electromechanical transduction factor (the "turns ratio") is ϕ.

I. READ FILES

Read configuration file:

$$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(ES2L\_config)$$

$a = 0.1$ $\quad$ $b = 0$ $\quad$ $tc = 1 \cdot 10^{-4}$

Read shape-integral file:

$$(Ip0\ \ Ip1\ \ Iq\ \ Iv\ \ Ike\ \ Ia\ \ wnmax\ \ N\ \ CC) := READPRN(ES2L\_deflect)$$

$$ac1 := 1 + 2\cdot\frac{z0}{tc} \qquad ac2 := 1 + 3\cdot\frac{z0}{tc} + 3\cdot\left(\frac{z0}{tc}\right)^2 \qquad as2 := 1 - 3\cdot\frac{z0}{tsub} + 3\cdot\left(\frac{z0}{tsub}\right)^2$$

$$Aeff := \pi\cdot a^2\cdot Ia \qquad Aeff = 0.031$$

II. CALCULATE EQUIVALENT-CIRCUIT PARAMETERS

Equivalent mass:

$$mass := 2\cdot\pi\cdot a^2\cdot Ike\cdot\frac{Ia^2}{Iv^2}\cdot(\rho c\cdot tc + \rho sub\cdot tsub) \qquad mass = 3.82$$

Equivalent stiffness:

$$f11 := cd11 - 0.75\cdot\varepsilon s33\cdot h31^2\cdot\frac{ac1^2}{ac2} \qquad f12 := cd12 - 0.75\cdot\varepsilon s33\cdot h31^2\cdot\frac{ac1^2}{ac2}$$

$$cs11 := \frac{Esub}{1 - vsub^2} \qquad cs12 := vsub\cdot cs11$$

$$Iff := f11\cdot Ip0 + f12\cdot Ip1 \qquad Iss := cs11\cdot Ip0 + cs12\cdot Ip1$$

$$stiffness := \frac{2\cdot\pi}{3\cdot a^2}\cdot\frac{Ia^2}{Iv^2}\cdot\left(tc^3\cdot ac2\cdot Iff + tsub^3\cdot as2\cdot Iss\right) \qquad stiffness = 1.16\cdot 10^8$$

Transduction factor:

$$\phi := -\left(\pi\cdot h31\cdot\varepsilon s33\cdot Iq\cdot\frac{Ia}{Iv}\cdot tc\cdot ac1\right) \qquad \phi = 1.461$$

Blocked electrical capacitance:

$$CEB := \frac{\varepsilon s33\cdot\pi\cdot a^2}{tc}\cdot Ia \qquad CEB = 2.482\cdot 10^{-6}$$

Coupling factor:

$$kf1 := stiffness\cdot\frac{Iv^2}{Ia^2} \qquad kf2 := \pi\cdot\varepsilon s33\cdot h31^2\cdot tc^3\cdot\frac{Iq^2}{Ia}\cdot\frac{ac1^2}{a^2}$$

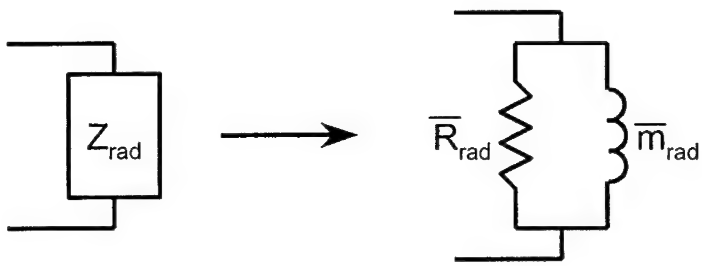
$$\kappa 2 := \frac{kf2}{kf1 + kf2} \qquad \kappa 2 = 7.352 \cdot 10^{-3}$$

Calculate equivalent Q for composite structure:

$$Uratio := \left(\frac{tsub}{tc}\right)^3 \cdot \frac{as2}{ac2} \cdot \frac{Iss}{Iff} \qquad Qeff := Qmech \cdot (1 + Uratio)$$

$$Rmech := \sqrt{\frac{stiffness}{mass}} \cdot \frac{mass}{Qeff}$$

Calculate circuit elements for parallel form of radiation impedance. This form is preferred over the series form since, in the parallel form, the resistance element is frequency independent. Furthermore, the parallel form degrades more gracefully for ka approaching (or greater than) one. Also, the assumption is made that these transducer elements would always be used in pairs placed back-to-back and driven in phase. In this drive mode, the radiation impedance is equivalent to that of a single element in an infinite rigid baffle.



$$\rho\_water := 1000 \qquad c\_water := 1500$$

$$m\_rad := \frac{8}{3 \cdot \pi} \cdot \rho\_water \cdot a \cdot \left(\pi \cdot a^2\right) \cdot Ia \qquad R\_rad := 1.44 \cdot \rho\_water \cdot c\_water \cdot \left(\pi \cdot a^2\right) \cdot Ia$$

Write equivalent-circuit parameters to a file:

$$WRITEPRN(ES2L\_equivckt) := (mass \ \ stiffness \ \ \phi \ \ CEB \ \ \kappa 2 \ \ \tan\delta \ \ Rmech \ \ R\_rad \ \ m\_rad \ \ Aeff)$$

III. MISCELLANEOUS CALCULATIONS

Unloaded resonance frequency:

$$mech\_res := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{stiffness}{mass}} \qquad mech\_res = 877.156$$

Rough estimate of water loading (piston with uniform face velocity):

$$\text{rad\_mass} := \frac{8}{3\cdot\pi}\cdot\rho\_\text{water}\cdot\pi\cdot a^3\cdot \text{Ia} \qquad \omega2\text{water} := \frac{\text{stiffness}}{\text{mass} + \text{rad\_mass}} \qquad \frac{\rho\_\text{water}\cdot a}{\rho c\cdot 2\cdot \text{tc}} = 65.789$$

$$\text{mech\_res\_water} := \frac{1}{2\cdot\pi}\cdot\sqrt{\omega2\text{water}} \qquad \text{mech\_res\_water} = 673.113$$

$$\text{wavenum} := \frac{2\cdot\pi\cdot\text{mech\_res\_water}}{\text{c\_water}} \qquad \text{ka} := \text{wavenum}\cdot a \qquad \text{ka} = 0.282$$

$$\text{radR} := \rho\_\text{water}\cdot\text{c\_water}\cdot\pi\cdot a^2\cdot\text{Ia}\cdot\frac{\text{ka}^2}{2} \qquad \text{radR} = 1.873\cdot 10^3$$

Estimated maximum source level (if strain limited):

[Note: Maximum strain is not an appropriate failure criterion for a ceramic material. It is used in this draft version as a crude estimate of limiting performance. Eventually, the more appropriate maximum tensile stress criterion will be implemented. The often-used von Mises stress criterion is also inappropriate for failure in ceramic.]

Enter maximum allowable strain $\text{Smax} := 0.001$

$$\text{Qmax} := \pi\cdot a^2\cdot 2\cdot\pi\cdot\text{mech\_res\_water}\cdot\text{wnmax}\cdot\text{Smax}\cdot\text{Iv}$$

$$\text{p1} := \left|\rho\_\text{water}\cdot\text{mech\_res\_water}\cdot\text{Qmax}\right| \qquad \text{SL} := 20\cdot\log(\text{p1}) + 120$$

Maximum pressure at 1 meter (Pa): $\text{p1} = 2.92\cdot 10^4$

Maximum SL in dB with respect to 1 micropascal at one meter: $\text{SL} = 209.3$

Estimated peak TVR:

What follows is an optimistic estimate of the peak transmitting voltage response; mechanical loss in the ceramic is ignored.

$$\text{tvr\_peak} := \rho\_\text{water}\cdot\text{mech\_res\_water}\cdot\pi\cdot a^2\cdot\text{Ia}\cdot\frac{\phi}{\text{radR}}$$

TVR in pascals at one meter per volt input: $\text{tvr\_peak} = 16.489$

TVR in dB with respect to 1 micropascal at one meter per volt: $20\cdot\log(\left|\text{tvr\_peak}\right|) + 120 = 144.3$

Low-frequency receive response:

The receiving response is not normally important but can be useful in reciprocity calculations.

$$\mathrm{ffvs} := \frac{\pi \cdot a^2 \cdot Ia}{\phi} \cdot \left(\frac{1}{1 + \frac{CEB \cdot stiffness}{\phi^2}} \right)$$

FFVS in open-circuit volts per pascal: $\mathrm{ffvs} = 1.581 \cdot 10^{-4}$

FFVS in dB with respect to one volt per micropascal: $20 \cdot \log(|\mathrm{ffvs}|) - 120 = -196.0$

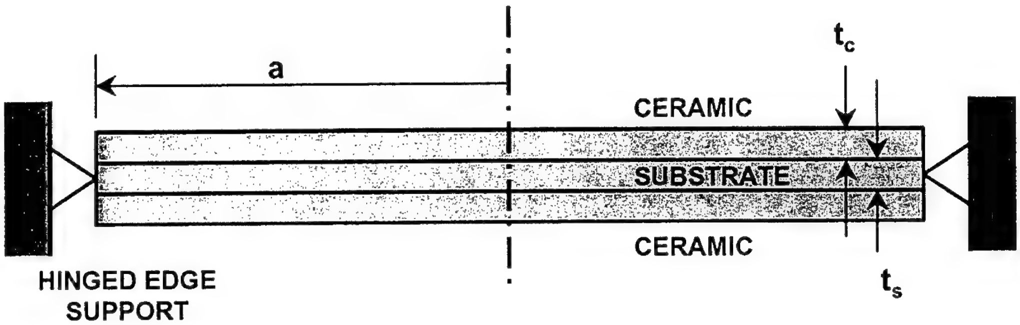
The quantity in square brackets in the equation for ffvs should equal 2:

$$\frac{1}{1 + \frac{CEB \cdot stiffness}{\phi^2}} = 7.352 \cdot 10^{-3} \qquad \kappa 2 = 7.352 \cdot 10^{-3}$$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 1.ES3L_config: Configuration Definition - Edge-Supported Trilaminar Disk

This worksheet is specific to the three-layer edge-supported disk. First, the physical dimensions are specified and the piezoelectric properties file is read. Then, the two-dimensional properties are derived. Finally, the results are written to a configuration file.



I. SELECT MATERIALS [Highlighted values are user inputs]

Choose from PZT4, PZT5H, or PZT8 for file name in pzt READPRN.

$\text{pzt} := \text{READPRN(PZT4)}$

$\text{se11} := \text{pzt}_{0,0} \cdot 10^{-12} \cdot \frac{1}{\text{Pa}}$ $\text{se12} := \text{pzt}_{0,1} \cdot 10^{-12} \cdot \frac{1}{\text{Pa}}$ $\varepsilon\text{t33} = \text{pzt}_{0,2} \cdot \varepsilon 0$

$\text{d31} := \text{pzt}_{0,3} \cdot 10^{-12} \cdot \frac{\text{m}}{\text{volt}}$ $\rho := \text{pzt}_{0,4} \cdot \frac{\text{kg}}{\text{m}^3}$ $\tan\delta := \text{pzt}_{0,5}$ $\text{Qmech} := \text{pzt}_{0,6}$

Choose from steel, aluminum, or brass for file name in substrate READPRN.

$\text{substrate} := \text{READPRN(brass)}$

$\rho\text{sub} := \text{substrate}_{0,0} \cdot \frac{\text{kg}}{\text{m}^3}$ $\text{Esub} := \text{substrate}_{0,1} \cdot 10^9 \cdot \text{Pa}$ $\text{vsub} := \text{substrate}_{0,2}$

$\rho\text{sub} = 8.5 \cdot 10^3 \ \cdot \text{kg} \cdot \text{m}^{-3}$ $\text{Esub} = 1.04 \cdot 10^{11} \ \cdot \text{kg} \cdot \text{m}^{-1} \cdot \text{sec}^{-2}$ $\text{vsub} = 0.37$

II. SELECT DIMENSIONS

Enter dimensions of ceramic disk (a = outside radius; tc = thickness)
and thickness (tsub) of substrate disk:

[Enter units for in, mm, cm, or m; if no units are entered, meters will be assumed]

$a := 10\cdot cm$ $\quad tc := 1\cdot mm$ $\quad tsub := 1\cdot cm$ $\quad b := 0$

$a = 0.1\cdot m$ $\quad b = 0$ $\quad tc = 1\cdot 10^{-3}\cdot m$ $\quad ts2 := \dfrac{tsub}{2}$

III. DERIVE PROPERTIES FOR TWO-DIMENSIONAL ANALYSIS

$$sd11 := se11 - \frac{d31^2}{\varepsilon t33} \qquad sd12 := se12 - \frac{d31^2}{\varepsilon t33} \qquad -\frac{sd12}{sd11} = 0.485$$

$$cd11 := \frac{sd11}{sd11^2 - sd12^2} \qquad cd12 := \frac{-sd12}{sd11^2 - sd12^2} \qquad \frac{cd12}{cd11} = 0.485$$

$$\varepsilon s33 := \varepsilon t33 - \frac{2\cdot d31^2}{se11 + se12} \qquad h31 := \frac{d31}{\varepsilon t33\cdot(sd11 + sd12)}$$

IV. WRITE CONFIGURATION FILE

PRN files cannot handle dimensions so all quantities are written (in SI) without units.

$$out_{0,0} := \frac{\rho}{\rho 0} \qquad out_{0,1} := \frac{a}{m0} \qquad out_{0,2} := b \qquad out_{0,3} := \frac{tc}{m0}$$

$$out_{0,4} := \frac{cd11}{c0} \qquad out_{0,5} := \frac{cd12}{c0} \qquad out_{0,6} := \frac{h31}{h0} \qquad out_{0,7} := \frac{\varepsilon s33}{e0}$$

$$out_{0,8} := \tan\delta \qquad out_{0,9} := Qmech \qquad out_{0,10} := \frac{\rho sub}{\rho 0} \qquad out_{0,11} := \frac{Esub}{c0}$$

$$out_{0,12} := vsub \qquad out_{0,13} := \frac{ts2}{m0} \qquad out_{0,14} := \frac{ts2}{m0}$$

$WRITEPRN(ES3L\_config) := out$

The configuration file contains the following quantities in this order:

density of ceramic, ρ
outer radius of disk, a
ratio of inner radius to outer radius, b (always zero here)
thickness of ceramic, tc
cd11 for ceramic
cd12 for ceramic
h31 for ceramic
εs33 for ceramic
loss tangent for ceramic, tanδ
mechanical Q of ceramic
density of substrate, ρsub
modulus of substrate, Esub
poisson's ratio of substrate, vsub
half thickness of substrate, ts2
location of neutral plane (same as half substrate thickness)

Unit normalization quantities (predefined):

$\rho 0 \equiv 1 \cdot \frac{kg}{m^3}$ $\quad m0 \equiv 1 \cdot m$ $\quad c0 \equiv 1 \cdot Pa$ $\quad h0 \equiv 1 \cdot \frac{volt}{m}$ $\quad e0 \equiv 1 \cdot \frac{farad}{m}$

$\varepsilon 0 \equiv 8.854 \cdot 10^{-12} \cdot \frac{farad}{m}$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 2.ES3L_deflect: Edge-Supported Trilaminar Disk

This worksheet reads the configuration file, calculates the first axisymmetric mode deflection function, and then calculates all of the shape integrals. The shape integrals are written to a file for subsequent use. In this version, the polynomial Rayleigh-Ritz method is used to compute the mode deflection function. An eighth-order polynomial is used. This module is expecting the configuration file from version 2 -- that is, the configuration with arbitrary location of the neutral plane and specific properties for the substrate.

I. READ CONFIGURATION FILE

$(\rho c\ \ a\ \ b\ \ tc\ \ cd11\ \ cd12\ \ h31\ \ \varepsilon s33\ \ \tan\delta\ \ Qmech\ \ \rho sub\ \ Esub\ \ vsub\ \ tsub\ \ z0) := \mathrm{READPRN}(\mathrm{ES3L\_config})$

$a = 0.1$ $\quad b = 0$ $\quad tc = 0.001$ $\quad vc := \dfrac{cd12}{cd11}$ $\quad vc = 0.485282$

$v := 0.5\cdot(vc + vsub)$ $\quad v = 0.427641$

II. MODE SOLUTION

In this section, the Rayleigh-Ritz technique is used to find the fundamental eigenvalue and eigenfunction for an 8th order polynomial. (N is preset to 8 below.)

$UU := U(N, b, v)$ $\quad TT := T(N, b, v)$

$gvals := \mathrm{genvals}(UU, TT)$

$gvecs := \mathrm{genvecs}(UU, TT)$

$sgcomb := \mathrm{rsort}\left(\mathrm{stack}\left(gvals^T, gvecs\right), 0\right)$

$fundval := \sqrt{sgcomb_{0,0}}$ $\quad fundval = 5.116155$

III. FIND THE COEFFICIENTS FOR THE DEFLECTION FUNCTION

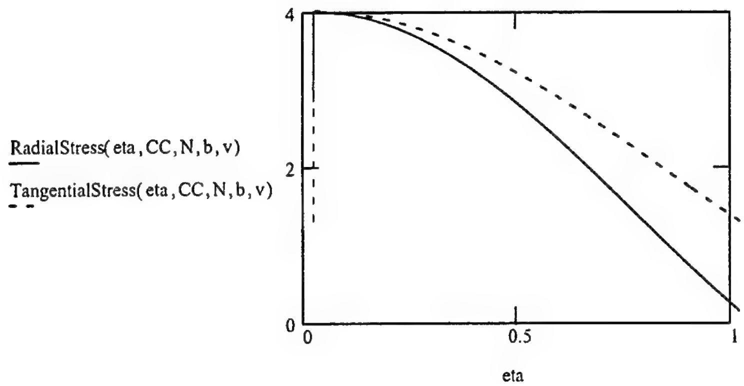
$fundcoef := \mathrm{submatrix}(sgcomb, 1, N, 0, 0)$

$CC := \mathrm{scaledcoef}(fundcoef, N, b)$

The coefficients are normalized so that the maximum deflection is one.

IV. PLOT THE NORMALIZED STRESSES

$\text{eta} := b, b + 0.002 .. 1$



V. CALCULATE THE SHAPE INTEGRALS

$\text{Ike} := CC^T \cdot TT \cdot CC$	$\text{Ike} = 0.140926$
$\text{Ia} := 1 - b^2$	$\text{Ia} = 1$
$\text{Iv} := CC^T \cdot VV(N, b)$	$\text{Iv} = 0.445882$
$\text{Iq} := CC^T \cdot QQ(N, b)$	$\text{Iq} = 1.401826$
$\text{Ipsum} := CC^T \cdot \text{Usum}(N, b, v) \cdot CC$	$\text{Ipsum} = 4.813486$
$\text{Ip1} := CC^T \cdot \text{Up1}(N, b, v) \cdot CC$	$\text{Ip1} = 1.965116$
$\text{Ip0} := \text{Ipsum} - \text{Ip1}$	$\text{Ip0} = 2.84837$

The maximum strain is at the inner radius. The second derivative of the normalized deflection is calculated below (maxcurve) and the peak deflection (i.e., deflection at r = a) corresponding to a strain of one is saved (wnmax).

$$\text{maxcurve} := CC^T \cdot SM(N, b) \qquad \text{wnmax} := \frac{a^2}{(tc + z0) \cdot \text{maxcurve}_0} \qquad \text{wnmax} = 0.616393$$

Write shape-integral file:

$$\text{WRITEPRN}(\text{ES3L\_deflect}) := \left(\text{Ip0}_0 \;\; \text{Ip1}_0 \;\; \text{Iq}_0 \;\; \text{Iv}_0 \;\; \text{Ike}_0 \;\; \text{Ia} \;\; \text{wnmax} \;\; N \;\; CC\right)$$

Because several of the shape integrals are calculated through matrix operations, they remain as matrices even though having dimension 1x1. The subscript index is used to convert to scalar form before writing. The coefficients are written as a matrix (Nx1) as CC.

What follows are predefinitions...

$N \equiv 8$

$$\text{scaledcoef}(vect, N, b) \equiv \left| \begin{array}{l} peakdefl \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad peakdefl \leftarrow peakdefl + vect_{ii-1} \\ vectscaled \leftarrow \dfrac{vect}{peakdefl} \\ vectscaled \end{array} \right.$$

$$\text{Usum}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad ua_{m-1,n-1} \leftarrow \dfrac{(m+1)^2 \cdot (n+1)^2}{m-n} \\ ua \end{array} \right.$$

$$\text{Up1}(N, b, v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad ubb_{m-1,n-1} \leftarrow \dfrac{n \cdot (n+1) \cdot (m+1)}{m+n} \\ ub \leftarrow ubb + ubb^T \\ ub \end{array} \right.$$

$$U(N, b, v) \equiv \left| \begin{array}{l} uu \leftarrow \text{Usum}(N, b, v) - (1-v) \cdot \text{Up1}(N, b, v) \\ uu \end{array} \right.$$

$$\text{SM}(N, b) \equiv \left| \begin{array}{l} \text{for } m \in 2..N \\ \quad sm_{m-1} \leftarrow 0 \\ sm_0 \leftarrow 2 \\ sm \end{array} \right.$$

$$T(N,b,v) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad \text{for } n \in 1..N \\ \quad\quad tt_{m-1,n-1} \leftarrow \dfrac{1}{m+n+4} - \dfrac{1}{m+3} - \dfrac{1}{n+3} + \dfrac{1}{2} \\ tt \end{array} \right.$$

$$VV(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow 1 - \dfrac{2}{m+3} \\ matv \end{array} \right.$$

$$QQ(N,b) \equiv \left| \begin{array}{l} \text{for } m \in 1..N \\ \quad matv_{m-1} \leftarrow (m+1) \\ matv \end{array} \right.$$

$$slope(eta, vect, N, b) \equiv \left| \begin{array}{l} slope \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad slope \leftarrow slope + vect_{ii-1} \cdot (ii+1) \cdot eta^{ii} \\ slope \end{array} \right.$$

$$curve(eta, vect, N, b) \equiv \left| \begin{array}{l} curve \leftarrow 0 \\ \text{for } ii \in 1..N \\ \quad curve \leftarrow curve + vect_{ii-1} \cdot (ii+1) \cdot ii \cdot eta^{ii-1} \\ curve \end{array} \right.$$

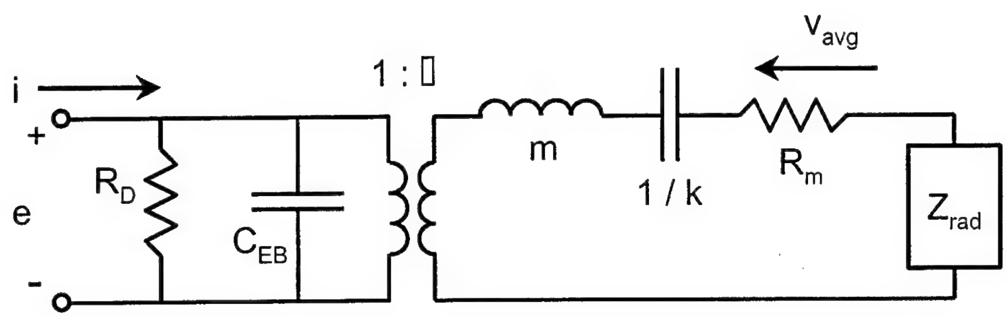
$$RadialStress(eta, vect, N, b, v) \equiv curve(eta, vect, N, b) + \frac{v \cdot slope(eta, vect, N, b)}{eta}$$

$$TangentialStress(eta, vect, N, b, v) \equiv v \cdot curve(eta, vect, N, b) + \frac{slope(eta, vect, N, b)}{eta}$$

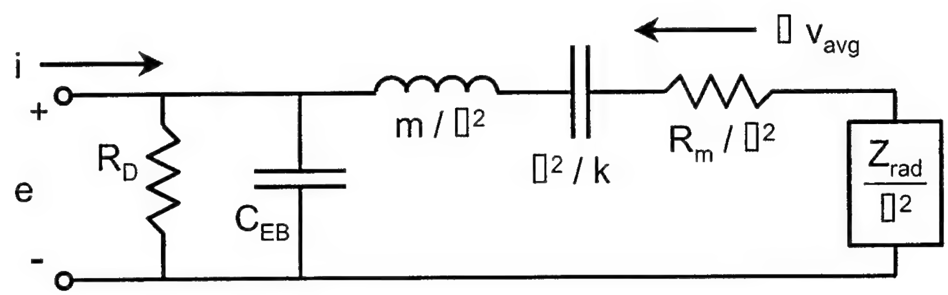
Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Module 3: Equivalent-Circuit - Trilaminar, Edge-Supported Flex Disks

This worksheet reads the configuration and shape-integral files for a particular case and computes the equivalent-circuit parameters.



Alternate form:



In these equivalent circuits, the mechanical quantities, force (pA or acoustic pressure times effective area) and velocity, v, are related to the electrical quantities, voltage, e, and current, i. C_{EB} is the blocked electrical capacitance and R_D is the dielectric loss ($\tan\delta/\omega*C_{EB}$). The mechanical mass, stiffness, and mechanical damping are m, k, and R_m respectively. The mechanical resistance is computed based on the mechanical Q of the ceramic and the fraction of strain energy that is stored in the ceramic layer of the composite. The radiation impedance, Zrad, is represented as a parallel combination of resistance and mass (see below). Finally, the electromechanical transduction factor (the "turns ratio") is ϕ.

I. READ FILES

Read configuration file:

$$(\rho c \;\; a \;\; b \;\; tc \;\; cd11 \;\; cd12 \;\; h31 \;\; \varepsilon s33 \;\; \tan\delta \;\; Qmech \;\; \rho sub \;\; Esub \;\; vsub \;\; tsub \;\; z0) := \mathrm{READPRN}(ES3L\_config)$$

$a = 0.1$ $\qquad$ $b = 0$ $\qquad$ $tc = 1 \cdot 10^{-3}$

Read shape-integral file:

$$(\mathrm{Ip0}\ \ \mathrm{Ip1}\ \ \mathrm{Iq}\ \ \mathrm{Iv}\ \ \mathrm{Ike}\ \ \mathrm{Ia}\ \ \mathrm{wnmax}\ \ \mathrm{N}\ \ \mathrm{CC}) := \mathrm{READPRN(ES3L\_deflect)}$$

$$\mathrm{ac1} := 1 + 2\cdot\frac{\mathrm{z0}}{\mathrm{tc}} \qquad \mathrm{ac2} := 1 + 3\cdot\frac{\mathrm{z0}}{\mathrm{tc}} + 3\cdot\left(\frac{\mathrm{z0}}{\mathrm{tc}}\right)^2 \qquad \mathrm{as2} := 1 - 3\cdot\frac{\mathrm{z0}}{\mathrm{tsub}} + 3\cdot\left(\frac{\mathrm{z0}}{\mathrm{tsub}}\right)^2$$

$$\mathrm{Aeff} := \pi\cdot a^2\cdot\mathrm{Ia} \qquad \mathrm{Aeff} = 0.031$$

II. CALCULATE EQUIVALENT-CIRCUIT PARAMETERS

Equivalent mass:

$$\mathrm{mass} := 4\cdot\pi\cdot a^2\cdot\mathrm{Ike}\cdot\frac{\mathrm{Ia}^2}{\mathrm{Iv}^2}\cdot(\rho c\cdot\mathrm{tc} + \rho\mathrm{sub}\cdot\mathrm{tsub}) \qquad \mathrm{mass} = 4.463$$

Equivalent stiffness:

$$\mathrm{f11} := \mathrm{cd11} - 0.75\cdot\varepsilon\mathrm{s33}\cdot\mathrm{h31}^2\cdot\frac{\mathrm{ac1}^2}{\mathrm{ac2}} \qquad \mathrm{f12} := \mathrm{cd12} - 0.75\cdot\varepsilon\mathrm{s33}\cdot\mathrm{h31}^2\cdot\frac{\mathrm{ac1}^2}{\mathrm{ac2}}$$

$$\mathrm{cs11} := \frac{\mathrm{Esub}}{1 - \mathrm{vsub}^2} \qquad \mathrm{cs12} := \mathrm{vsub}\cdot\mathrm{cs11}$$

$$\mathrm{Iff} := \mathrm{f11}\cdot\mathrm{Ip0} + \mathrm{f12}\cdot\mathrm{Ip1} \qquad \mathrm{Iss} := \mathrm{cs11}\cdot\mathrm{Ip0} + \mathrm{cs12}\cdot\mathrm{Ip1}$$

$$\mathrm{stiffness} := \frac{4\cdot\pi}{3\cdot a^2}\cdot\frac{\mathrm{Ia}^2}{\mathrm{Iv}^2}\cdot\left(\mathrm{tc}^3\cdot\mathrm{ac2}\cdot\mathrm{Iff} + \mathrm{tsub}^3\cdot\mathrm{as2}\cdot\mathrm{Iss}\right) \qquad \mathrm{stiffness} = 1.747\cdot 10^8$$

Transduction factor for parallel connection of ceramic layers:

$$\phi := -2\cdot\left(\pi\cdot\mathrm{h31}\cdot\varepsilon\mathrm{s33}\cdot\mathrm{Iq}\cdot\frac{\mathrm{Ia}}{\mathrm{Iv}}\cdot\mathrm{tc}\cdot\mathrm{ac1}\right) \qquad \phi = 3.213$$

Blocked electrical capacitance:

$$\mathrm{CEB} := \frac{2\cdot\left(\varepsilon\mathrm{s33}\cdot\pi\cdot a^2\right)}{\mathrm{tc}}\cdot\mathrm{Ia} \qquad \mathrm{CEB} = 4.965\cdot 10^{-7}$$

Coupling factor for parallel connection of ceramic layers:

$$\mathrm{kf1} := \mathrm{stiffness}\cdot\frac{\mathrm{Iv}^2}{\mathrm{Ia}^2} \qquad \mathrm{kf2} := 2\cdot\pi\cdot\varepsilon\mathrm{s33}\cdot\mathrm{h31}^2\cdot\mathrm{tc}^3\cdot\frac{\mathrm{Iq}^2}{\mathrm{Ia}}\cdot\frac{\mathrm{ac1}^2}{a^2}$$

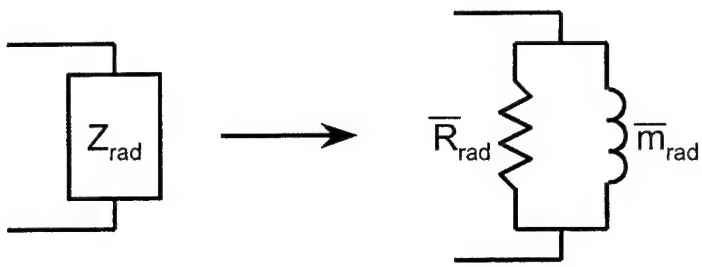
$$\kappa 2 := \frac{kf2}{kf1 + kf2} \qquad \kappa 2 = 0.106$$

Calculate equivalent Q for composite structure:

$$Uratio := \left(\frac{tsub}{tc}\right)^3 \cdot \frac{as2}{ac2} \cdot \frac{Iss}{Iff} \qquad Qeff := Qmech \cdot (1 + Uratio)$$

$$Rmech := \sqrt{\frac{stiffness}{mass}} \cdot \frac{mass}{Qeff}$$

Calculate circuit elements for parallel form of radiation impedance. This form is preferred over the series form since, in the parallel form, the resistance element is frequency independent. Furthermore, the parallel form degrades more gracefully for ka approaching (or greater than) one. Also, the assumption is made that these transducer elements would always be used in pairs placed back-to-back and driven in phase. In this drive mode, the radiation impedance is equivalent to that of a single element in an infinite rigid baffle.



$$\rho\_water := 1000 \qquad c\_water := 1500$$

$$m\_rad := \frac{8}{3 \cdot \pi} \cdot \rho\_water \cdot a \cdot \left(\pi \cdot a^2\right) \cdot Ia \qquad R\_rad := 1.44 \cdot \rho\_water \cdot c\_water \cdot \left(\pi \cdot a^2\right) \cdot Ia$$

Write equivalent-circuit parameters to a file:

$$WRITEPRN(ES3L\_equivckt) := (mass \;\; stiffness \;\; \phi \;\; CEB \;\; \kappa 2 \;\; \tan\delta \;\; Rmech \;\; R\_rad \;\; m\_rad \;\; Aeff)$$

III. MISCELLANEOUS CALCULATIONS

Unloaded resonance frequency:

$$mech\_res := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{stiffness}{mass}} \qquad mech\_res = 995.732$$

Rough estimate of water loading (piston with uniform face velocity):

$$rad\_mass := \frac{8}{3\cdot\pi}\cdot\rho\_water\cdot\pi\cdot a^3\cdot Ia \qquad \omega2water := \frac{stiffness}{mass + rad\_mass} \qquad \frac{\rho\_water\cdot a}{\rho c\cdot 2\cdot tc} = 6.579$$

$$mech\_res\_water := \frac{1}{2\cdot\pi}\cdot\sqrt{\omega2water} \qquad mech\_res\_water = 787.8$$

$$wavenum := \frac{2\cdot\pi\cdot mech\_res\_water}{c\_water} \qquad ka := wavenum\cdot a \qquad ka = 0.33$$

$$radR := \rho\_water\cdot c\_water\cdot\pi\cdot a^2\cdot Ia\cdot\frac{ka^2}{2} \qquad radR = 2.566\cdot 10^3$$

Estimated maximum source level (if strain limited):

[Note: Maximum strain is not an appropriate failure criterion for a ceramic material. It is used in this draft version as a crude estimate of limiting performance. Eventually, the more appropriate maximum tensile stress criterion will be implemented. The often-used von Mises stress criterion is also inappropriate for failure in ceramic.]

Enter maximum allowable strain $Smax := 0.001$

$$Qmax := \pi\cdot a^2\cdot 2\cdot\pi\cdot mech\_res\_water\cdot wnmax\cdot Smax\cdot Iv$$

$$p1 := \left|\rho\_water\cdot mech\_res\_water\cdot Qmax\right| \qquad SL := 20\cdot\log(p1) + 120$$

Maximum pressure at 1 meter (Pa): $p1 = 3.367\cdot 10^4$

Maximum SL in dB with respect to 1 micropascal at one meter: $SL = 210.5$

Estimated peak TVR:

What follows is an optimistic estimate of the peak transmitting voltage response; mechanical loss in the ceramic is ignored.

$$tvr\_peak := \rho\_water\cdot mech\_res\_water\cdot\pi\cdot a^2\cdot Ia\cdot\frac{\phi}{radR}$$

TVR in pascals at one meter per volt input: $tvr\_peak = 30.989$

TVR in dB with respect to 1 micropascal at one meter per volt: $20\cdot\log(\left|tvr\_peak\right|) + 120 = 149.8$

Low-frequency receive response:

The receiving response is not normally important but can be useful in reciprocity calculations.

$$ffvs := \frac{\pi \cdot a^2 \cdot Ia}{\phi} \cdot \left(\frac{1}{1 + \frac{CEB \cdot stiffness}{\phi^2}} \right)$$

FFVS in open-circuit volts per pascal: $ffvs = 1.04 \cdot 10^{-3}$

FFVS in dB with respect to one volt per micropascal: $20 \cdot \log(|ffvs|) - 120 = -179.7$

The quantity in square brackets in the equation for ffvs should equal ▯2:

$$\frac{1}{1 + \frac{CEB \cdot stiffness}{\phi^2}} = 0.106 \qquad \kappa 2 = 0.106$$

Flexural-Disk Transducer Analytical Model - DRAFT VERSION 3.0

Performance Module

This worksheet can be used with any of the flexural-disk configurations. Several performance-related measures are calculated from the basic equivalent-circuit representation. These include: (1) unloaded admittance, (2) water-loaded admittance, (3) transmitting voltage response, and (4) transmitting current response. For (3) and (4) the radiation impedance is taken to be that of a piston in an infinite, rigid baffle. This is appropriate either for a truly baffled transducer or for a transducer that was constructed from TWO flexural-disk elements mounted back-to-back as would normally be done in a high-performance system. The two-element system is the image equivalent of a single projector in an infinite, rigid baffle. Remember that the electrical characteristics (e.g., (1) and (2) above) are for a single element. If a two-element system is being modeled, the user must adjust the electrical parameters according to the electrical connection of the elements (series or parallel).

Required user input is the name of the file containing the equivalent-circuit parameters:

$$(\text{mass} \;\; \text{stiffness} \;\; \phi \;\; \text{CEB} \;\; \kappa 2 \;\; \tan\delta \;\; \text{Rmech} \;\; \text{R\_rad} \;\; \text{m\_rad} \;\; \text{Aeff}) := \text{READPRN}(\text{CC2L\_equivckt})$$

Restore physical SI units:

$\text{mass} := \text{mass}\cdot\text{mass\_unit}$ $\quad\quad$ $\text{stiffness} := \text{stiffness}\cdot\text{stiffness\_unit}$

$\phi := \phi\cdot\phi\text{\_unit}$ $\quad\quad$ $\text{CEB} := \text{CEB}\cdot\text{capacitance\_unit}$

$\text{Rmech} := \text{Rmech}\cdot\text{Rmech\_unit}$ $\quad\quad$ $\text{R\_rad} := \text{R\_rad}\cdot\text{Rmech\_unit}$

$\text{m\_rad} := \text{m\_rad}\cdot\text{mass\_unit}$ $\quad\quad$ $\text{Aeff} := \text{Aeff}\cdot\text{area\_unit}$

If you want to model a closed, fluid-filled cavity behind the flexural disk, set 'cavity' equal to one; if not, make sure that 'cavity' is set to zero and skip entry of cavity/fluid properties.

$\text{cavity} := 0$

Enter optional fluid-cavity properties (set 'cavity' to zero to disable

Enter volume, vc, of cavity; density, ρf, of cavity fluid; and sound speed, cf, of cavity fluid:

[For a back-to-back two element transducer, enter only half the total cavity volume.]

$\text{vc} := 0.0016\cdot\text{m}^3$ $\quad\quad$ $\rho\text{f} := 800\cdot\dfrac{\text{kg}}{\text{m}^3}$ $\quad\quad$ $\text{cf} := 1400\cdot\dfrac{\text{m}}{\text{sec}}$

[Make sure to enter units for quantities; if no units are entered, SI will be assumed]

Cavity stiffness:

$$kc := \rho f \cdot cf^2 \cdot \frac{(Aeff)^2}{vc}$$

$$stiffness = 4.157 \cdot 10^7 \cdot kg \cdot sec^{-2}$$
$$kc = 2.229 \cdot 10^8 \cdot kg \cdot sec^{-2}$$

$$stiffness := stiffness + kc \cdot cavity$$

The unloaded resonance frequency and Q are

$$f0 := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{stiffness}{mass}} \qquad f0 = 1.027 \cdot 10^3 \cdot sec^{-1}$$

$$Q\_unload := 2 \cdot \pi \cdot f0 \cdot \frac{mass}{Rmech} \qquad Q\_unload = 1.146 \cdot 10^3$$

The electrical equivalents of the mechanical mass, compliance, and resistance are

$$mass\_e := \frac{mass}{\phi^2} \qquad Cm\_e := \frac{\phi^2}{stiffness} \qquad Rmech\_e := \frac{Rmech}{\phi^2}$$

The admittance calculation is broken into three parts: Y1 is the dielectric loss, Y2 is the blocked capacitance, and Y3 is the electrical equivalent of the mechanical section.

$$Y1(w) := w \cdot CEB \cdot \tan\delta \qquad Y2(w) := j \cdot w \cdot CEB \qquad Y3(w) := \frac{1}{Rmech\_e + j \cdot w \cdot mass\_e + \frac{1}{j \cdot w \cdot Cm\_e}}$$

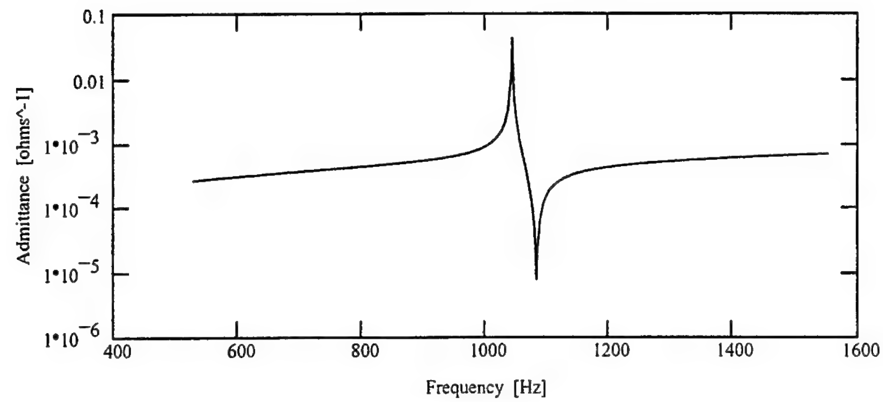
$$Y\_unloaded(f) := Y1(2 \cdot \pi \cdot f) + Y2(2 \cdot \pi \cdot f) + Y3(2 \cdot \pi \cdot f)$$

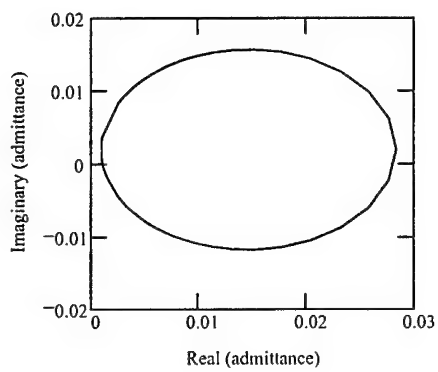
The awkward part of plotting the unloaded admittance is that the unloaded Q is high and the resonance peak is sharp. The fmat routine produces a matrix (fx) of unevenly spaced frequencies so that the resonance is well defined in the plot without excessive calculation of points away from the resonance. For water-loaded calculations, this is unnecessary.

$$fx := fmat(f0, Q\_unload, \kappa 2) \qquad nn := 0 .. length(fx) - 1$$

$$ymat(fx) := \left| \begin{array}{l} \text{for } mm \in 0 .. length(fx) - 1 \\ \quad yy_{mm} \leftarrow Y\_unloaded\left(fx_{mm}\right) \\ yy \end{array} \right. \qquad Y\_U := ymat(fx)$$

UNLOADED ADMITTANCE





Calculate the water-loaded resonance frequency and the Q:

$$f0w := \frac{1}{2\cdot\pi}\cdot\sqrt{\frac{stiffness}{mass + m\_rad}} \qquad xx := \left(\frac{2\cdot\pi\cdot f0w\cdot m\_rad}{R\_rad}\right)^2 \qquad R\_at\_f0\_rad := R\_rad\cdot\left(\frac{xx}{1+xx}\right)$$

$$Q\_water := 2\cdot\pi\cdot f0\cdot\frac{mass + m\_rad}{Rmech + R\_at\_f0\_rad} \qquad f0w = 745.664\cdot sec^{-1} \qquad Q\_water = 22.728$$

Calculate the electrical equivalents of the radiation impedance terms and calculate the electrical equivalent of the radiation impedance (using the parallel mass/resistance model):

$$mrad\_e := \frac{m\_rad}{\phi^2} \qquad Rrad\_e := \frac{R\_rad}{\phi^2} \qquad Zrad\_e(w) := \frac{j\cdot w\cdot Rrad\_e\cdot mrad\_e}{Rrad\_e + j\cdot w\cdot mrad\_e}$$

Calculate the admittance term for the mechanical side including the radiation load:

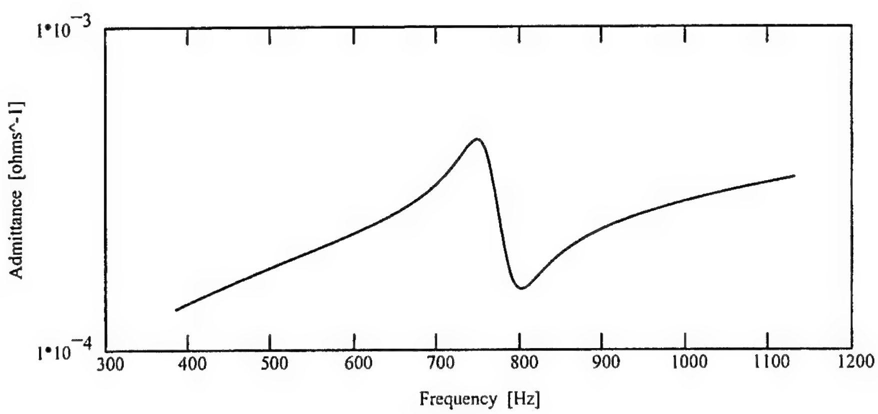
$$Y4(w) := \frac{1}{Rmech\_e + j \cdot w \cdot mass\_e + \dfrac{1}{j \cdot w \cdot Cm\_e} + Zrad\_e(w)}$$

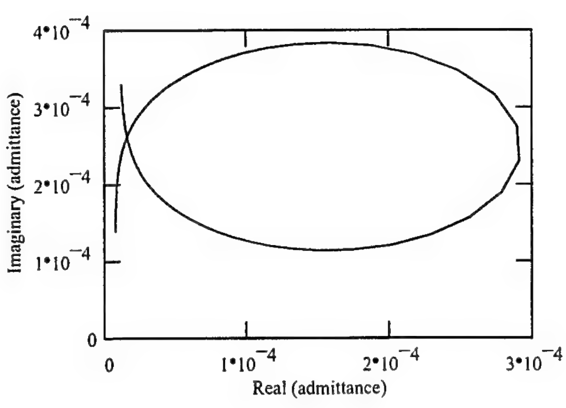
Calculate and plot the total admittance. An even increment in frequency is suitable since the Q should be much lower with the water load.

$$Y\_water(f) := Y1(2 \cdot \pi \cdot f) + Y2(2 \cdot \pi \cdot f) + Y4(2 \cdot \pi \cdot f)$$

$$ff1 := 0.5 \cdot f0w \qquad ff2 := 1.5 \cdot f0w \qquad dff := \frac{ff2 - ff1}{200} \qquad ffx := ff1, ff1 + dff .. ff2$$

WATER-LOADED ADMITTANCE





Calculation of Transmitting Voltage Response

The pressure (in Pa) at one meter per volt drive is

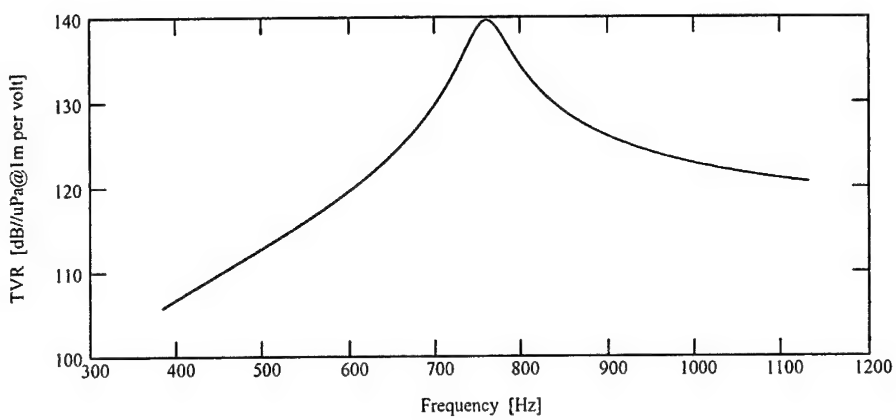
$$\text{rho\_water} := 1000 \cdot \frac{kg}{m^3} \qquad \text{p\_per\_volt}(f) := \text{Aeff} \cdot \text{rho\_water} \cdot f \cdot \frac{Y4(2 \cdot \pi \cdot f)}{\phi}$$

or, in dB with respect to one micropascal at one meter per volt,

$$\text{dB\_TVR\_ref} := 10^{-6} \cdot \frac{Pa}{volt} \cdot 1 \cdot m$$

$$\text{dB\_TVR}(f) := 20 \cdot \log\left(\frac{\text{p\_per\_volt}(f)}{\text{dB\_TVR\_ref}}\right)$$

TRANSMITTING VOLTAGE RESPONSE



Calculation of Transmitting Current Response

The pressure (in Pa) at one meter per ampere drive is

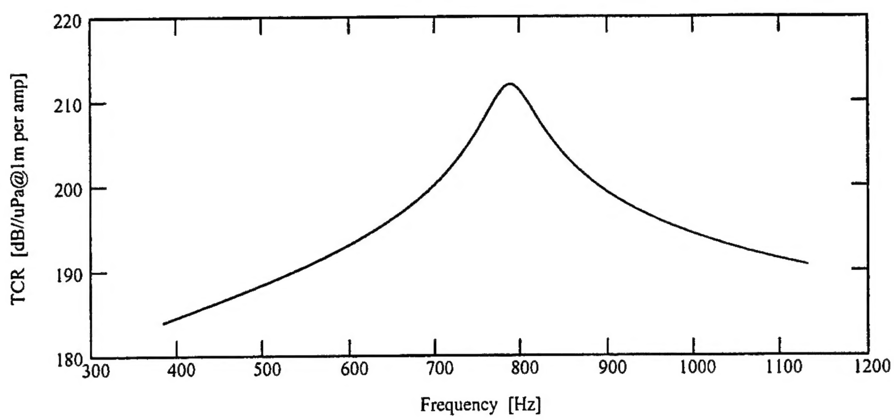
$$\text{p\_per\_amp}(f) := \frac{\text{p\_per\_volt}(f)}{\text{Y\_water}(f)}$$

or, in dB with respect to one micropascal at one meter per ampere,

$$\text{dB\_TCR\_ref} := 10^{-6} \cdot \frac{Pa}{amp} \cdot 1 \cdot m$$

$$\text{dB\_TCR}(f) := 20 \cdot \log\left(\frac{\text{p\_per\_amp}(f)}{\text{dB\_TCR\_ref}}\right)$$

TRANSMITTING CURRENT RESPONSE



Predefinition of routine to calculate frequencies for unloaded admittance plot:

$$
\mathrm{fmat}(f0,Q,k2) \equiv \left|
\begin{array}{l}
f1 \leftarrow 0.5 \\
f2 \leftarrow 0.6 \\
fdip \leftarrow \dfrac{1}{\sqrt{1-k2}} \\
dpm \leftarrow fdip - 1 \\
f3 \leftarrow 1 + 0.375 \cdot dpm \\
f4 \leftarrow 1 + 0.625 \cdot dpm \\
f5 \leftarrow 1.1 \cdot fdip \\
f6 \leftarrow 1.2 \cdot fdip \\
fom \leftarrow 1 - \dfrac{2}{Q} \\
fop \leftarrow 1 + \dfrac{2}{Q} \\
fdipm \leftarrow fdip - \dfrac{2}{Q} \\
fdipp \leftarrow fdip + \dfrac{2}{Q} \\
f5 \leftarrow \mathrm{if}(f5 < 1.4, 1.4, f5) \\
f6 \leftarrow \mathrm{if}(f6 < 1.5, 1.5, f6) \\
f2 \leftarrow \mathrm{if}(f2 < fom, fom, f2) \\
f3 \leftarrow \mathrm{if}(f3 > fop, fop, f3) \\
f4 \leftarrow \mathrm{if}(f4 < fdipm, fdipm, f4) \\
f5 \leftarrow \mathrm{if}(f5 > fdipp, fdipp, f5) \\
dfa \leftarrow \dfrac{f2 - f1}{100}
\end{array}
\right.
$$

$$
\begin{aligned}
& cum\_m \leftarrow 0 \\
& \text{for } mm \in 0..99 \\
& \quad ff_{mm} \leftarrow f1 + mm \cdot dfa \\
& dfb \leftarrow \frac{f3 - f2}{50} \\
& cum\_m \leftarrow cum\_m + 100 \\
& \text{for } mm \in 0..49 \\
& \quad ff_{cum\_m + mm} \leftarrow f2 + mm \cdot dfb \\
& dfc \leftarrow \frac{f4 - f3}{25} \\
& cum\_m \leftarrow cum\_m + 50 \\
& \text{for } mm \in 0..24 \\
& \quad ff_{cum\_m + mm} \leftarrow f3 + mm \cdot dfc \\
& dfd \leftarrow \frac{f5 - f4}{50} \\
& cum\_m \leftarrow cum\_m + 25 \\
& \text{for } mm \in 0..49 \\
& \quad ff_{cum\_m + mm} \leftarrow f4 + mm \cdot dfd \\
& dfe \leftarrow \frac{f6 - f5}{100} \\
& cum\_m \leftarrow cum\_m + 50 \\
& \text{for } mm \in 0..99 \\
& \quad ff_{cum\_m + mm} \leftarrow f5 + mm \cdot dfe \\
& ff \cdot f0
\end{aligned}
$$

Predefinitions of unit factors

$mass\_unit \equiv 1 \cdot kg$

$Rmech\_unit \equiv 1 \cdot \frac{newton \cdot sec}{m}$

$stiffness\_unit \equiv 1 \cdot \frac{newton}{m}$

$area\_unit \equiv 1 \cdot m^2$

$\phi\_unit \equiv 1 \cdot \frac{newton}{volt}$

$capacitance\_unit \equiv 1 \cdot farad$

Appendix I. MathCad Materials Files

PZT4.prn

12.3	-4.05	1300	-122	7600	0.004	500

PZT5H.prn

16.5	-4.78	3400	-274	7500	0.02	70

PZT8.prn

11.5	-3.7	1000	-97	7500	0.004	1050

brass.prn

8500	104	0.37

aluminum.prn

2700	70	0.33

steel.prn

7700	195	0.28

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